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Children's Use of the Reference Point Strategy for Measurement Estimation

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Mathematics educators frequently recommend that students use strategies for measurement estimation, such as the reference point or benchmark strategy; however, little is known about the effects of using this strategy on estimation accuracy or representations of standard measurement units. One reason for the paucity of research in this area is that students rarely make use of this strategy spontaneously. In order to boost students' strategy use so that we could investigate the relationships among strategy use, accuracy of students' representations of standard measurement units, and estimation accuracy, 22 third-grade students received instruction on use of the reference point strategy and another 22 third-grade students received instruction on the guess-and-check procedure. Analyses reveal that children's strategy use predicts the accuracy of their representations of standard linear measurement units and their estimates. Relative to students who did not use a reference point, students who used a reference point had more accurate representations of standard units and estimates of length.

Key words: Children's strategies; Elementary K–8; Estimation; Measurement; Number sense; Visualization/Spatial reasoning

Understanding how to estimate measurements is important for two reasons. First, it is an essential numeracy skill in its own right and is often called upon in daily life when measuring instruments are absent or inconvenient (Levin, 1981; O'Daffer, 1979). Second, and perhaps more important, measurement estimation provides a convenient route for teaching physical measurement—a central topic in the elementary mathematics curriculum that forms the foundation for other mathematical concepts such as fractions and ratio (Coburn & Shulte, 1986; Davydov & Tsvetkovich, 1991; National Council of Teachers of Mathematics [NCTM], 2000). It is not surprising then that NCTM has repeatedly recommended measurement estimation as a main focus for the K–5 curriculum (NCTM, 1989, 2000).

The estimation of linear measurements is deceptively simple: In many ways, it seems similar to counting except that instead of counting discrete objects, one counts the number of units into which an object is divided (Joram, Subrahmanyam, & Gelman, 1998). For example, when estimating the length of a pen, one may divide the total length of the pen into inches and count the number of inches. Several developmental psychologists (e.g., Gelman, 1991) have suggested that counting does not appear to present conceptual difficulties for most children, and that most children learn to count discrete objects with relative ease. Why is it then, with its apparent isomorphism to counting, we find that American students typically perform poorly on all but the most straightforward of measurement estimation tasks (Beaton, Mullis, Martin, Gonzalez, Kelly, & Smith, 1996; Sowder, 1992)?

We have argued that it is the division of an object into units that makes measurement estimation a much more formidable task than it may at first seem (Joram et al., 1998). The most common strategy used for measurement estimation is to "mentally measure" an object (Friebel, 1967; Hartley, 1977; Hildreth, 1983; Immers, 1983). Mental measurement, or "unit iteration," entails recalling an image of a standard unit such as a foot, repeatedly comparing that image to the object that is to be estimated, keeping track of where the last unit ended and the next one should begin, and maintaining a running tally of the units while continuing to perform the tasks above. Researchers have found that unit iteration is the strategy that estimators default to in the absence of formal instruction on other strategies for measurement estimation (Friebel, 1967; Hartley, 1977; Hildreth, 1983; Immers, 1983), with 30% to 97% of estimators using this strategy.

Estimators may make use of a variety of other estimation strategies that serve to reduce some of the cognitive demands otherwise present when using the unit iteration strategy (Friebel, 1967; Hartley, 1977; Hildreth, 1983; Immers, 1983; Sowder, 1992). For example, the *reference point*[1] or *benchmark* strategy (Carter, 1986; Hildreth, 1983; Joram, 2003; Spitzer, 1961) involves imagining an object whose measurement is known (e.g., a paper clip known to be 1-inch long), and comparing it with the to-be-estimated object (e.g., the length of a pen). Sowder (1992) notes that in order to form an estimate, "one must have a mental reference unit, that is, a mental 'picture' or 'feel' for the size of the unit" (p. 371).

Because reference points typically consist of objects familiar to the estimator, they may be more meaningful than the corresponding standard unit (Carter, 1986), and should, therefore, assist estimators in recalling the magnitude of a standard unit. For example, when a dermatologist suggests that one pay attention to moles bigger than 6 mm (about the size of a pencil eraser), the reference point "pencil eraser"

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[1] The term *reference point* denotes an object to which an estimator can psychologically connect a measurement unit or multiple units rather than a point in space.

will probably be more meaningful than the measurement 6 mm because it is familiar and therefore more easily represented. Clements and McMillen (1996) note that mathematical ideas are meaningful by virtue of their connections to other ideas and situations—by providing a bridge between a number and a familiar quantity, reference points can make measurement units more meaningful to estimators.

A reference point may be similar in magnitude to a standard unit (e.g., a paper clip or one's thumb from fingertip to knuckle to represent 1 inch; one side of a linoleum tile to represent 1 foot), or a multiple of a standard unit (e.g., the length of a pen to represent 6 inches; one's own height, which is typically about five or six times the standard unit of a foot). When making use of a reference point that is similar in magnitude to a standard unit, the reference point must be iterated if the to-be-estimated object is greater in magnitude than one reference point "unit." Using a reference point that is a multiple of a standard unit may further facilitate the process of measurement estimation by reducing the number of unit iterations that must be performed. For example, instead of subdividing a room into 12 feet and counting them to estimate its length, one need only divide the room into approximately two-person lengths and count these. Doubling the known length of one's own height (e.g., approximately 6 feet), yields an estimate of 12 feet for the length of the room. In such cases, by decreasing the total number of unit iterations that need to be performed, use of the reference point strategy reduces the frequency with which the estimator needs to keep track of where one unit ends and the next begins, and the total number of units tallied. Of course, the to-be-estimated object is frequently not an exact multiple of the reference point, and a final adjustment needs to be made in this kind of situation (e.g., estimating the room to be the equivalent of two person-lengths plus a couple of feet). Other examples of combination strategies, in which the reference point strategy is used in conjunction with an additional strategy, can be found in Joram et al. (1998).

Thus, reference points have two advantages over their corresponding standard units: (1) they make measurements units more meaningful, and (2) they may reduce the total number of unit iterations that must be performed mentally, thereby reducing error. We suggest that there is a third advantage that reference points may offer, especially to young estimators: Because reference points that stand for a single standard unit consist of a discrete object of a particular magnitude, the process of unit iteration and the constraints that operate on that process may be more salient to the estimator than when imagining a unit on a standard measuring instrument. Lehrer, Jaslow, and Curtis (2003) note that:

> One of the important ideas about a unit of length measure is that the measure is obtained by first subdividing a length and then repeatedly translating or "iterating" this subdivision. Although both these characteristics of unit may seem obvious to all users of rulers, they often are not apparent to young children, even those who are proficient at using rulers. . . . (p. 102)

For teachers of measurement, reference points provide an opportune way to illustrate the correct process of iterating a unit, including honoring the constraints of equal unit size, and continuity of units (no spaces can be left between units and the space must be completely filled with units). Although rulers and yardsticks embody these constraints, as Lehrer et al. (2003) point out, students may not fully apprehend these properties, even when skilled at using these tools. Reference points, on the other hand, allow students to first physically, and then mentally, model the process of unit iteration.

MEASUREMENT ESTIMATION STRATEGIES AND ESTIMATION ACCURACY

Although mathematics educators have assumed a beneficial role for the use of strategies in estimating measurements, very few studies have demonstrated that use of these strategies is related to greater estimation accuracy. Previous researchers have typically assessed either estimation accuracy (e.g., Siegel, Goldsmith, & Madson, 1982), or estimation strategy use (e.g., Friebel, 1967), but have rarely assessed accuracy and strategy use in the same study. On those occasions when separate measures of measurement estimation strategy use and accuracy have been included in the same study, either accuracy has not been analyzed as a function of strategy use (e.g., Forrester, Latham, & Shire, 1990), or researchers have included as many as eight different strategies under the umbrella term "strategy use" and because of this, were not able to identify which strategies were associated with gains in estimation accuracy (e.g., Forrester & Shire, 1994; Hildreth, 1983).

Reference Point Strategy

To our knowledge, no studies have been conducted that have permitted an examination of the relationship between use of the reference point strategy and estimation accuracy. One obstacle to investigating this relationship is that students do not spontaneously use this strategy very often, as Hildreth's (1983) study indicated. Hildreth did not find that many students used the reference point strategy, or similar strategies, very frequently before or after instruction. He did not include the reference point strategy in the repertoire of strategies he taught fifth- and seventh-grade students so it is not surprising that they failed to use this strategy after instruction. In the study reported in this article, we instructed students how to use reference points for measurement estimation in hopes of boosting the frequency of their strategy use; we would then be in a position to investigate how the use of this strategy is related to accuracy of estimates and representations of standard units, such as an inch (*unit representations*). In addition, we documented errors that students made when using strategies so that we could better interpret information collected about students' estimation strategy use and estimation accuracy.

Unit Iteration

We were unable to locate studies that examined the relationship of the unit iteration strategy and estimation accuracy. Hildreth (1983) included unit iteration among those strategies he taught students and found a small increase in estimation accuracy as a result of instruction or practice. However, as noted above, Hildreth taught eight different strategies in his instructional unit, making it impossible to attribute gains in estimation accuracy to the use of any particular strategy.

INSTRUCTION ON MEASUREMENT ESTIMATION

Although the literature indicates that the unit iteration strategy is the preferred strategy for estimating measurements, we were not able to locate studies that instructed students on the use of this strategy. Our brief literature review is, therefore, limited to summarizing studies on the two most common methods of teaching measurement estimation: the reference point strategy and the guess-and-check procedure.

Reference Point Strategy

The reference point strategy, described above, has been frequently recommended by mathematics educators as a way to teach measurement estimation (Bright, 1976; NCTM, 1989, 2000; Spitzer, 1961; Thompson & Rathmell, 1989), both because it presumably makes measurement units more meaningful and because of its potential to facilitate the process of estimating, as discussed above. As also discussed above, few studies have investigated the efficacy of any strategy for measurement estimation, and the reference point strategy is no exception (Joram et al., 1998; Sowder, 1992). We located only two studies examining the use of the reference point strategy (Attivo & Trueblood, 1980; Joram, Gabriele, Gelman, & Subrahmanyam, 1996), and both suffered from methodological limitations. Only one of these studies, which we describe next, involved school-aged children.

Findings from a study by Joram et al. (1996) are consistent with the notion that the reference point method may enhance measurement estimation for elementary level students. Third-grade students, who were taught to imagine reference points that were equivalent to an inch and a foot, drew more accurate representations of the corresponding standard units in a posttest than a group of students who had practiced guessing and checking the accuracy of estimates of measurements. Although the researchers found that estimation accuracy improved significantly more for the reference point group on a few objects (e.g., pieces of rope of different lengths), they did not find this effect when students estimated the dimensions of a variety of real-world objects that varied more in their shape than the rope—for example, the length of one side of a videocassette case. That is, when objects were not long and narrow, students were less accurate estimating, perhaps because they had difficulty focusing on only the dimension of length. Notwithstanding the limited findings of this study, it is consistent with the notion that the reference point strategy may support the processes involved in estimating measurements for students.

Guess-and-Check Procedure

In addition to instructing students on the use of strategies for measurement estimation, mathematics educators and teachers have often simply asked students to guess the measurement of a given dimension of an object (e.g., length of a pencil), and then to check their guess using a measurement instrument such as a ruler. Introduced in the 1920s, the guess-and-check procedure has maintained its popularity, as an examination of current textbook series in elementary mathematics demonstrates (e.g., Harcourt School Publishers, 2002; Silver Burdett Ginn, 2001). According to the behaviorist theory from which this method was derived (e.g., Thorndike, 1927; Trowbridge & Cason, 1932), checking provides corrective feedback, which results in subsequent estimates becoming more accurate. Later psychologists (Gibson & Bergman, 1954; Gibson, Bergman, & Purdy, 1955) argued that use of the guess-and-check procedure is limited, and that estimators learn only associations between specific measurements and their corresponding objects. Similar to the reference point strategy, the use of guess and check has produced only limited gains in estimation accuracy (Joram et al., 1998; Sowder, 1992).

When estimators guess and then check the measurements of objects or extents, we do not know what they are actually doing cognitively when estimating—this is why we call guess and check a "procedure" here rather than a "strategy." In other words, an estimator who guesses and then checks his or her estimate of a measurement could be using any one of a number of strategies or no strategy at all. For example, estimators could iterate a standard unit in order to come up with their guess, or they could remember the lengths of previously guessed and checked objects and compare the to-be-estimated object with this repertoire of remembered and checked objects. It is, therefore, of interest to examine what strategies are used by individuals who have practiced guessing and checking.

RESEARCH QUESTIONS

Our first set of research questions pertained to what strategies third-grade students would use (if any) spontaneously, and the accuracy of their unit representations and estimates prior to strategy intervention. Further, we were interested in the relationship between students' use of the reference point strategy and other aspects of measurement estimation competency. On the basis of pilot work and published literature, we predicted that the students in our study would infrequently make use of the reference point strategy for measurement estimation; by providing instruction, we intended to increase the frequency of the use of this strategy among students so that we could investigate these research questions:

1. Are the estimates of linear measurements (using inch and foot units) of those students who use reference points more accurate than those students who do not?
2. Are the unit representations of those students who use reference points more accurate than those students who do not?
3. What kinds of errors do students make when iterating both reference points and standard units?

To develop the methods employed in this study, we drew on the published literature on teaching measurement estimation using reference points (Joram et al., 1996). For pragmatic reasons, one intact class was taught the reference point method, and another class was given experience in guessing and checking measurements. The literature on measurement estimation has shown that iteration of a standard unit is the default strategy for estimators in absence of instruction on a specific strategy. Thus, we anticipated that students in the guess-and-check group would rely, to a great extent, on the iteration of a standard unit when estimating, and in doing so they would form a convenient comparison group. Providing the comparison group with experiences similar to the reference point group helped to insure that differences potentially revealed after instruction would not be due to students' differential expectations or other spurious factors such as their unfamiliarity with the task of estimating.

METHOD

Participants

Forty-four third-grade students (23 girls and 21 boys) from two intact classes participated in this study. All students were attending a school in an urban midwestern city that serves primarily low-income families. Fifty percent of the students were members of ethnic minority groups, primarily African American, and none of the participants were categorized as having limited proficiency in English. Seventy-six percent of the students in the school received a free or reduced lunch program.

The *reference point group* consisted of 22 students from one class, and the *guess-and-check group* of 22 students was from another class; within the two classes, gender was almost evenly distributed. At the beginning of the academic year in which the study was conducted, the third-grade students in the two classes scored at 2.6, and 2.5 grade equivalent levels in all mathematical concepts, placing them in the 32nd and 25th percentiles, respectively, on national norms for the Iowa Test of Basic Skills. Scores on the Iowa Test of Basic Skills on both measurement and estimation concepts revealed that the majority of students in both classes scored at or below the 25th percentile in these areas. Thus, at the beginning of the study, both classes were below an average level of mathematics achievement overall and were very similar in their mathematical achievement levels and their specific competency in related measurement and estimation concepts.

Design and Procedures

The teachers of students who participated in this study both reported that they taught mathematics in a similar, traditional style, relying mainly on a textbook (D. C. Heath, 1989) for seatwork and exercises on mathematical procedures and rarely using hands-on activities or group work in mathematics classes. These reports were corroborated by the researchers based on their observations of one of each teacher's mathematics lessons. By the time the study took place in January, neither class had begun to work on measurement; this was agreed upon by the researchers and the teachers at the beginning of the school year. The teachers were told that the researchers would be trying out some methods of teaching measurement.

Both the reference point and guess-and-check groups received six 45-minute mathematics lessons on linear estimation over 1 1/2 weeks, with group pretests and posttests given before and immediately after the instructional units. Both the reference point and guess-and-check lessons were taught during the regular mathematics period to the two classes consecutively by the third author of this article. She had participated in a 5-year professional development program that focused on reform-based mathematics instruction and had been recommended for the study by the leader of this program. This teacher had taught at the elementary level for approximately 30 years and collaborated with the other authors on the design of lessons for teaching measurement estimation.

In both the reference point and guess-and-check groups, estimating in units of inches and feet were covered. To simplify the description of the instruction below, most examples that are given in the sections that follow pertain to inches; the instruction was repeated for the foot unit for both groups. Yards were not focused on in instruction, but were introduced briefly to both groups.

Reference Point Strategy

The main focus of the reference point instruction was to help the students develop more accurate representations of linear measurement units, to show them how to correctly iterate a reference point to estimate, and to use reference points flexibly. Below, we describe the central components of the instruction.

Developing accurate representations of measurement units. Previous research with low-achieving elementary mathematics students reveals that most have inaccurate representations of standard linear units (Joram et al., 1996). Helping students in this study develop more accurate representations of measurement units was achieved through a variety of instructional activities. First, students made use of personal reference points for measurement units, an approach developed in a previous study (Joram et al., 1996). Students each chose a reference point to use from a set of "child-friendly" reference points (e.g., tiny plastic cars; pieces of bubble gum that measured approximately 1 inch in length). At the beginning of each lesson on inch units, students were asked to close their eyes, to imagine their own reference point, and then to "pick up an inch"—that is, to show it with their fingers or hands. Students placed their reference point between their partner's fingers to check the accuracy of the partner's physical representation, and the partner then corrected the size shown if necessary. Students also drew lines of various lengths and received feedback on the accuracy of their drawings. As a final way of helping students develop accurate representations of units, the teacher frequently drew lines on a white board, and the class chorally indicated, by saying "Stop!" when she had reached previously specified measurements, for example, 3 inches.

Although extensive use of reference points was made, students also used rulers and yardsticks to ensure that they would make connections between the reference points and the standard units shown on measuring instruments. Clements (1999) has argued that standard instruments introduced early in measurement instruction alongside nonstandard units may be of much greater benefit than the "traditional nonstandard-then-standard-then-ruler sequence" (p. 6).

Iterating reference points when estimating. Students were encouraged to think of reference points when estimating and practiced doing so, first physically and then mentally. The teacher demonstrated how to iterate a reference point, for example, by laying out a row of six pieces of bubble gum to measure the length of a pen. She later showed how just one piece could be used—repeatedly laying down the piece of gum while keeping track of where the last placement had ended. These demonstrations were repeated with inch-long magnets on a white board to facilitate students making the connection between the reference point units and standard units. Students practiced iterating a reference point to estimate, showing and iterating an imaginary inch with their fingers, and imagining themselves iterating their reference point to estimate.

Our intention was to help students move beyond manipulating physical objects to developing a mental picture of this process. We emphasized several of the characteristics of the unit discussed by Lehrer et al. (2003) and Hiebert (1981, 1984) during this aspect of the instruction: that spaces are not permitted between units, and that the length, width, or height of the entire to-be-estimated object/extent must be covered by units.

Using reference points flexibly. In order to encourage students to use reference points flexibly—that is, to use part of a reference point or a combination of reference points to estimate a measurement—students were shown how a familiar reference point could be divided in half or doubled. For example, a dollar bill was shown as a 6-inch reference point, and the teacher demonstrated how the dollar bill could be folded in half to yield a 3-inch reference point. Thus, the same reference point could be used to estimate a 3-inch long object (using a dollar bill folded in half), a 6-inch long object (a dollar bill), a 9-inch long object (a dollar bill plus folded dollar bill), or a foot-long object (2 dollar bills) by adding different combinations of the half or the whole reference point. In using reference points this way, we helped students see the additive relationships among units (e.g., 6 inches equals two 3-inch long reference points), and between different units (e.g., 1 foot equals two 6-inch long reference points). Objects were taped to a chart on the wall in order to highlight these relationships.

Guess-and-Check Procedure

The guess-and-check group completed many of the same or similar activities as the reference point group. For example, the first lesson for both groups began with the introductory activity of the teacher reading aloud the poem "If I Were One Inch Tall" (Silverstein, 1996), and asking each student to state one thing that would be different about their lives if they were only an inch tall. The focus for these students was on the use of standard instruments for measuring and estimating, the correct procedures for measuring, and on estimating accurately by receiving feedback on estimates.

The main difference between the two groups was that the reference point group received explicit instruction on *reference points*—imagining them, iterating reference points across objects both physically and mentally, and using them flexibly, as described above. The guess-and-check group, on the other hand, was simply asked to guess and check the measurements of the same group of objects that the reference point group had estimated. The students in this group were thus free to use strategies of their choosing. The students typically worked with partners and recorded their guesses and measurements on worksheets. To ensure that both the reference point and guess-and-check groups spent roughly the same amount of time working on measurement estimation, we additionally provided traditional instruction for the guess-and-check group that was based on material in their textbook. For example, when students in the reference point group were introduced to pieces of bubble gum as reference points for 1-inch units, students in the guess-and-check group were shown pictures in their textbook depicting a 1-inch long object such as a paper clip, and they examined inches on their rulers as well. They did not practice imagining or iterating representations of these inch-long objects, but completed exercises that involved inch units.

The guess-and-check group completed many exercises in their textbook that pertained to linear measurement. Thus, students filled out a number of worksheets in which they were required to measure the lengths of a variety of depicted objects such as feathers or pencils. Students additionally worked on a project taken from the enrichment pages of the teachers' manual corresponding to their textbook. In this exercise, each student created an ant farm on a large sheet of paper. The ant farm had to include several landmarks such as a post office, grocery store, and so on. Road signs had to be created that showed the distance from one landmark to the other (these were real distances that could be measured on the newsprint). We adapted this exercise to bring it in line with our goals for this group by having them work in pairs after their ant farm was completed, cover up their road signs showing distances with post-it notes, and have their partner guess and check the distances between landmarks.

Assessment Instruments: Individual Interviews

Each student was interviewed individually before and after strategy instruction and the guess-and-check procedure in order to classify and assess their estimation strategies and errors. Interviews were 15 to 20 minutes long and were conducted by either the first author or a research assistant. Students were asked to estimate the lengths of two objects while the interviewer recorded students' verbal responses and made notes on nonverbal behaviors that suggested the use of strategies. Although efforts were made to interview all students, one student was absent for the pretest interview; therefore, the sample size for the pretest interviews was 43. Two students were absent for the posttest interviews; thus, the sample size for the posttest interviews was 42 students.

In order to assess students' estimation strategies, during individual interviews each student was asked to estimate the lengths of two objects: a 9-inch long piece of rope and a 5-foot long piece of ribbon. Students were permitted to use their fingers and hands to point to imaginary units, but were not allowed to use a part of their body, such as their thumb, to actually measure the to-be-estimated object. Thus, students were not allowed to pace off units with their feet while estimating or to use any object such as a pen to help them make their estimate. Students' nonverbal behavior was noted and used to code strategies; for example, students who iterated units could often be seen to mark off the units by nods of their head or by pointing to unit divisions with their fingers above the object they were estimating.

In addition to inferring estimation strategies from nonverbal behaviors, students' self-reported estimation strategy use was also collected. Immediately after the student gave his or her estimate, and the interviewer had recorded both the answer and any nonverbal behavior indicating strategy use, the interviewer asked a series of questions beginning with "How did you come up with your answer?" If this question did not yield a response (e.g., if the student replied "I don't know" or said nothing), the interviewer continued: "What were you thinking about when you came up with your answer?" If nothing was forthcoming after this question, the interviewer would then ask: "Can you show me how you came up with your answer?" If a student, when giving their estimate, spontaneously provided enough information to address the first question, that student was not given any of the prompts above.

If a student responded that she or he had used a reference point (e.g., she or he might say "I was thinking about my cousin"), the interviewer would ask, "How did that help you come up with your answer?" If the student did not respond in such a way that indicated she or he knew the measurement of the reference point and had used this to arrive at the answer, the interviewer would further probe, "Do you know how tall your cousin is?" The rationale for this series of probes was that students would sometimes say that they had been thinking about an object when coming up with the estimate, but further questioning indicated that they did not use the object in a mathematical way. For example, the student might respond, "I was thinking about my jump rope at home," but when the interviewer then asked, "Do you know how long the jump rope is?" the student would respond, "No." The interviewer would then ask, "How did thinking about the jump rope help you come up with your answer?" and typically the student could not give an answer to this question. Such responses were not considered as indicating use of an appropriate reference point.

In order to examine the relationships among the use of the reference point strategy, estimation accuracy, and accuracy of unit representations, we distinguished among types of strategy users: (1) students who used a reference point to estimate one or both objects, (2) students who iterated a standard unit on at least one object and did not use a reference point to estimate either, and (3) students who used no identifiable strategy for either object. Interrater agreement was calculated on 50% of students' responses and was found to be 92%.

In addition to the categorization of errors above, we completed descriptive analyses of students' strategy use during the individual interviews that were conducted after the strategy intervention. Within the group of students who used a reference point, we tabulated the number of students who iterated that reference point to arrive at an estimate, and those who simply compared the reference point with the entire object. Within the group of students who did not use a reference point, we tabulated the number who iterated a standard unit and the number who guessed or used no identifiable strategy.

The interviewer also noted any errors in strategy use that the student made when iterating either a reference point or a standard unit. These included leaving spaces between units, using unequal-sized units, not keeping track of where one unit ended and the next began, and using an inaccurate-sized unit. An inaccurate-sized unit was defined as being 25% greater or less than the corresponding standard unit (e.g., < 0.75 or > 1.25 of 1 inch).

Assessment Instruments: Group Tests

In order to assess the accuracy of students' estimates and unit representations, group tests were administered to both classes several days before the instructional unit was given, as well as the day after. Students estimated the dimensions of a set of objects in inches and feet, and drew lines 1 inch and 1 foot in length. The group tests took approximately 45 minutes to complete, with students recording their responses in a booklet. The experimenter returned on subsequent days to administer the test to any students who were absent on the day of the test.

Estimation accuracy. On the group tests, students were asked to complete a series of six estimates of the length, height, or width of objects. Objects ranged in length from 3 inches to 9 feet and consisted mainly of everyday objects (e.g., length of a paintbrush, height of a door), none of which were used during instruction. The experimenter held up, or pointed to, each object in turn and the whole class wrote down their estimates. Students were told the unit they were to estimate with, for example, "How long is this crayon? Tell me in inches."

Students' estimates were typically given as whole numbers, with some fractional units, usually halves. Error scores were calculated for students' estimates.[2] The error score reflects the amount of deviation of a students' estimate from the target measurement, in either direction. For example, if a student estimated that the 5-foot long piece of ribbon was 10 feet in length, his or her error score would be 100%.

Average error scores were calculated for the estimates completed after strategy instruction and guess and check, and a logarithmic transformation[3] was performed

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[2] Error scores were computed through the following calculations: The absolute value of the difference of the students' line and the actual measurement of the line, divided by the actual measurement of the line. For example, if a student drew a line 8 inches long when asked to draw a line 1 foot long, the error score would be calculated as follows: $|8 - 12|/12 = 4/12 = 0.33$, or 33%.

[3] In this study, extreme scores appeared to result from students' lack of competence at drawing/estimating rather than from deviations from the study procedure, recording errors, etc. According to Kirk (1982), if a score deviates extremely from other observations but is considered to be an "extreme manifestation of the random variability inherent in data" (p. 139), rather than arising from procedural errors or other spurious factors, it should be retained. We therefore used a logarithmic transformation to reduce the variability otherwise present in the error scores rather than discarding extreme scores.

on the error scores to reduce variability in these scores. Untransformed scores are presented in tables for ease of interpretation, but analyses were performed on the log-transformed error scores. The same procedure was followed when analyzing the drawing data, as described below.

Accuracy of students' representations of standard units. To assess the accuracy of students' representations of measurement units, at the beginning of the group tests they were given sheets of paper 2 feet in length and were asked to draw line segments of 1 inch and 1 foot on a single side of the paper. To create an accuracy score for students' line drawings, each line was measured to the nearest 1/8 of an inch and an error score was calculated. Again, the error score reflected the amount of deviation of a students' line drawing from the target measurement, in either direction. For example, if a student was asked to draw a line measuring 1 foot in length, and he or she drew a line measuring 1 1/2 feet, his or her error score would be 50%.

RESULTS AND DISCUSSION

Students' Spontaneous Use of Estimation Strategies

Consistent with the literature indicating that students do not often spontaneously use the reference point strategy, the individual interviews revealed that, prior to strategy instruction, only 4 students (9%) out of all those we worked with used this strategy. The reference points that students mentioned included a fingertip, a pencil, a student's own height (which he said was "3 feet"), and the depth of a swimming pool. In contrast, 27 students (63%) iterated a standard unit. This pattern is consistent with other studies that have shown unit iteration to be the most common strategy for estimating (Friebel, 1967; Hartley, 1977; Hildreth, 1983; Immers, 1983). The remaining 12 students (28%) gave responses that were coded as using no identifiable strategy.

Accuracy of Students' Representations of Units and Estimates Prior to Strategy Intervention

As shown by the untransformed error scores in Table 1, prior to instruction on the reference point strategy and guess and check, students' drawings of an inch and a foot, made during the group tests, deviated approximately 60% to 85% from the target measurement. This finding confirmed our suspicions that most low-achieving third-grade students do not have accurate representations of standard units. Similarly, as shown in Table 1, the untransformed error scores for students' estimates on the group test were also quite high (ranging from about 94% to 136%), which is not surprising, given that two thirds of students used the iteration strategy but with an inaccurate unit representation. Across the three different strategy types we coded (including responses that showed no strategy), the students seemed to be uniformly weak in their unit representations and estimations of the lengths of objects.

Table 1
Mean Untransformed Error Scores of Unit Representations and Estimates by Strategy Use Prior to Strategy Intervention

		Task	
Strategy used during interviews	*n*	Unit representations (Group test)	Estimates (Group test)
Reference point	4	61.45 (46.63)	106.42 (89.13)
Iteration of standard unit	27	84.46 (97.43)	93.84 (75.74)
No strategy	12	72.82 (57.33)	136.14 (56.18)

Note. Lower mean error scores represent more accurate unit representations. Standard deviations appear in parentheses.

Students' Use of Estimation Strategies After Strategy Intervention

Many more students were identified as using the reference point strâtegy when estimating linear measurements after strategy instruction than before. Approximately two thirds of students (64%) who had received the strategy instruction used a reference point to estimate one or both of the objects in the posttest interviews. Of the remaining students who received the strategy instruction, 25% iterated a standard unit and only one student did not use an identifiable strategy when estimating. Interestingly, 14% of those students who were instructed to guess and check also made use of the reference point strategy. It is possible that these students may have spontaneously invented this strategy on their own or they may have borrowed the idea from friends who were in the reference point class. Fifty-five percent of students in the guess-and- check group iterated a standard unit, and 32% used no identifiable strategy.

Students who iterated a benchmark to estimate the 9-inch long piece of rope reported thinking about many of the objects that had been used in instruction—for example, pieces of bubble gum and tiny toy cars. A student who iterated a reference point to estimate the length of the 5-foot long piece of ribbon said that he imagined a strip of foam that was 1-foot long and iterated it in the air. He then showed 2 hand spans, saying that he thought that each one was 6 inches, and showed how he iterated these to come up with his estimate.

Other students recalled a benchmark representing multiple units and compared it with the rope or ribbon. For example, when estimating the 9-inch long piece of rope, one student said he was "Thinking about something at home, a Vortex—kind of a football, which is 7 inches long. I imagined it in my head and put it down by the rope." Another student stated that she was thinking about a Pringles can and "trying to measure in my mind against the rope." When estimating the 5-foot long ribbon, a student who compared a benchmark representing multiple feet with the ribbon said, "That ribbon is about as long as I am tall; how tall I am or my brother is" [How did that help?] "Placed me on the ribbon, because I am 5 feet tall." Almost all students who used this strategy did not adjust for the difference between the length of the reference point and the to-be-estimated object's length. For example, several students who used an image of a 6-inch long dollar bill to estimate stated that the 9-inch long rope was 6 inches long.

As described above, we were successful at increasing the use of the reference point strategy through the strategy instruction we provided. This allowed us to examine the relationships between use of the reference point strategy and other competencies related to measurement estimation. For the analyses presented in the next section, we collapsed the data across type of intervention condition (strategy instruction vs. guess and check), and report results according to the strategies that students used. Analyses were conducted on those students who, after strategy instruction and the guess-and-check procedure, used a reference point to estimate and those students who did not use a reference point to estimate.

Errors Made When Iterating Nonstandard and Standard Units

After strategy instruction, within the group of students who used a reference point strategy to estimate (n = 17), approximately two thirds (n = 11) iterated their nonstandard unit (i.e., reference point) whereas the remaining one third compared a single nonstandard unit with the target object. Of those students who did not use the reference point strategy (n = 25), about the same proportion (n = 17) of students iterated, but with a standard unit.

When iterating either a reference point or standard unit, there are many opportunities to make errors that would lead to inaccurate estimates. We were interested in examining these errors to determine whether using a reference point, relative to a standard unit, results in significant reductions in errors when iterating. Therefore, we examined errors made by the subset of students who iterated a reference point or standard unit (about two thirds of students who used a reference point, and about two thirds of students who used a standard unit). These results, which appear in Table 2, reveal that most students who iterated either a standard unit or reference point made at least a single error, for example, leaving spaces between units. There were only three students who, after strategy instruction and guess and check, estimated without making any of the errors shown on Table 2. The most common errors that students made when estimating were using an inaccurate-sized or inconsistent-sized unit and not keeping track of division points.

It is surprising that a similar number of students who both iterated a standard and nonstandard unit (i.e., reference point) made errors related to their representation of the unit, that is, inaccurate unit size and inconsistent unit size. This finding may partially be explained by the fact that our scoring was very strict for these errors, as described above. Thus, a student who used a reference point could be classified as using an inaccurate-sized unit while still estimating more accurately than a student who did not use a reference point.

Table 2
Errors Made by Students Who Iterated Reference Point or Standard Unit Based on Interviews After Strategy Intervention

Type of error	Group: Students who iterated reference point (n = 11)	Group: Students who iterated standard unit (n = 17)
Unit size inaccurate by more than 0.25 of the unit (e.g., < 0.75, > 1.25 of one inch)	6 (55%)	9 (53%)
Leaves spaces between units	1 (9%)	5 (29%)
Does not keep track of division points	9 (82%)	15 (88%)
Unit size inconsistent	2 (18%)	5 (29%)

Accuracy of Unit Representations and Estimates after Strategy Intervention

Table 3 shows the mean error scores for unit representations and estimates from the group test according to the strategy students used. Analyses indicated that there were no statistically significant differences in the accuracy scores of unit representations or estimates of students who iterated a standard unit and those who used no strategy, and therefore these two groups were collapsed for further analyses.

A t test on the log-transformed error scores for students' drawings of standard units, completed during the group test, indicated that after strategy instruction and practice guessing and checking, students who used reference points drew lines representing an inch and a foot with significantly greater accuracy than those students who did not use a reference point, $t(40) = 2.17$, $p < .05$. Looking at Table 3, it is interesting to note that the standard deviations of the untransformed estimation error scores for those students using a reference point are much smaller than for students

Table 3
Mean Untransformed Error Scores of Unit Representations and Estimates by Strategy Use After Strategy Intervention

Strategy used during interviews	n	Task: Unit representations (Group test)	Task: Estimates (Group test)
Reference point	17	24.90 (12.23)	32.75 (12.79)
Iteration of standard unit	17	48.25 (59.77)	99.01 (142.81)
No strategy	8	60.57 (32.84)	116.41 (84.51)

Note. Lower error scores represent more accurate unit representations and estimates. Although error scores are presented in this table for ease of interpretation, analyses reported in the article were performed on log-transformed scores in order to reduce the amount of variability in the error scores.

who did not, suggesting greater consistency in producing accurate drawings (e.g., $SD = 12.23$ for students using reference points, $SD = 59.77$ for those iterating a standard unit).

A similar pattern of results was found with the assessment of students' estimates made during the group test. A t test on the log-transformed error scores revealed that students who used the reference point strategy were more accurate at estimating linear measurements on a group paper-and-pencil test relative to those students who did not use this strategy, $t(40) = 2.72, p < .01$. Again, there is more constrained variability in the untransformed error scores of those students who used a reference point relative to those who did not use this strategy (e.g., $SD = 12.79$ for students using a reference point and $SD = 142.81$ for those iterating a standard unit). This suggests that, as a group, students using the reference point strategy more consistently gave "good" estimates. In contrast, there was much greater individual variability in the accuracy of estimates made by students who did not use the reference point strategy, indicating that some students executed their estimation strategies effectively whereas others had much less success.

Relationships Among Drawing Accuracy, Estimation Accuracy, and Strategy Use

Pearson product-moment correlation coefficients were calculated in order to examine the relationships among estimation accuracy, drawing accuracy, and strategy use. As shown in Table 4, all three variables we examined were significantly intercorrelated. Strategy use was found to be significantly related to both drawing and estimation accuracy: increased use of the reference point strategy is negatively correlated with amount of error on these measures. With respect to the relationship between students' drawing and estimation accuracy, the results are consistent with the hypothesis that students' line drawings reflect their representations of standard units, and that these representations come into play when students estimate measurements.

Table 4
Correlation Matrix for Drawing Accuracy, Estimation Accuracy, and Strategy Use

	1	2	3
1. Drawing accuracy	—	.57**	–.32*
2. Estimation accuracy		—	–.35*
3. Strategy use[a]			—

Note. Log-transformed data were used for the significance test.
[a] Strategy use coded as 0 = no reference point; 1 = reference point.
* Significant at the .05 level (2-tailed)
** Significant at the .01 level (2-tailed)

CONCLUSIONS

A number of mathematics educators (e.g., Markovits, Hershkowitz, & Bruckheimer, 1989; Sowder, 1992) have suggested that connecting numbers to referents in the context of measurement is an essential aspect of number sense in this domain. For example, Markovits et al. (1989) comment that "students know little of various everyday measurements that can serve as reference data for number sense in contextual problems, for example, the speed of a car, the height of an eight-story building . . ." (p. 54). They believe that students' tendency to respond with internally inconsistent answers to measurement estimation problems (e.g., stating that the width of a car is greater than the width of a road) is a direct result of students lacking such reference points. Similarly, Bright (1976) notes that good estimators have a well-developed repertoire of reference points that can be used flexibly.

Although mathematics educators have articulated their beliefs about the need for students to develop facility with the reference point strategy, there have been few empirical studies that have investigated the potential benefits of the use of this strategy. In our study, we found that the use of the reference point strategy was statistically associated with greater estimation accuracy both in the interview and group test settings. In addition, students who estimated using reference points had more accurate representations of standard linear units. Our findings suggest that, relative to students who do not use reference points to estimate, those students who use the reference point strategy have enhanced representations of linear units which leads to more accurate estimates. When standard units (or multiples of standard units) are represented by reference points, they seem to be more easily recalled and imagined than their corresponding standard units. Having a "feel" for standard linear units, represented by familiar objects, may form a foundation for two-dimensional measurements concepts (area), and three-dimensional measurement concepts (volume).

LIMITATIONS OF THE STUDY

The study reported in this article was conducted in one school with only 44 students. Because of the small sample size, any generalizations based on results of the study must be made with caution. Further, the students who participated in this study were low achievers in mathematics, and they began the instructional unit with little knowledge of linear measurement, as attested to by their scores on standardized tests. Thus, the finding that few students in this study spontaneously used a reference point may not be characteristic of other samples of third-grade students. Further, we do not know whether students who used reference points shortly after the instructional unit continued to do so several months later.

Finally, the design of this study precludes making any causal statements about the impact of instruction on reference points on the other variables investigated in this study. In future research, a well-designed instructional study in which use of reference points was taught in contrast to another strategy such as unit iteration would permit potential causal relationships between instruction and strategy use to be revealed.

Notwithstanding the limitations of this study, our results suggest that teaching students to use reference points for estimating is a promising area for future research.

Researchers might investigate the relationship between the use of strategies for measurement estimation and understanding of physical measurement. We can conjecture that a carefully designed unit of instruction on measurement estimation may enhance students' understanding of physical measurement. As pointed out in the introduction, when estimating measurements, students cannot rely on tools for measuring and, therefore, may not get as focused on the procedures of measuring (e.g., reading a number on a ruler) as when they physically measure an object or extent with a measuring instrument. Future researchers might explore how this shift in emphasis away from the procedures involved in measuring to the processes of estimating benefits students' understanding of physical measurement concepts.

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