PATHWAYS TO NUMBER
Children's Developing Numerical Abilities

Edited by
Jacqueline Bideaud
Université Charles de Gaulle, Lille III
Claire Meljac
Centre Henri-Rousselle, Paris
Jean-Paul Fischer
Institut Universitaire de Formation des Maîtres de Lorraine: Site de Montigny-lès-Metz

Selected chapters translated by
Constance Greenbaum

LEA
1992
LAWRENCE ERLBAUM ASSOCIATES, PUBLISHERS
Hillsdale, New Jersey
Hove and London

The Developmental Status of Number Conservation. There have been numerous attempts (e.g., Bryant, 1974; Gelman, 1982a; McGarrigle & Donaldson, 1975) to refute Piaget by showing that children have a basic understanding of number conservation, or an *invariance principle*, much earlier than he thought. The claim that children have a conservation operator for small numbers before they have one for large numbers is one type of early conservation claim. To contend with such claims, we need to ask where number conservation is located in the overall course of number development. We cannot maintain, as Piaget and Szeminska (1941) did, that conservation is the single true mark of having a "number concept." Children's quantifiers and operators develop considerably before conservation normally emerges. By the same token, we do not need to claim that they understand conservation from the start in order to validate young children's knowledge of number. We need to ask what problems a conservation operator would solve for the young child, and indeed how conservation gets identified as a problem in the first place. Perhaps children have to represent addition and subtraction transformations for large and small numbers before they seek a rule for what happens when nothing is added or subtracted (Cooper, 1984). To explain the development of number conservation, we need to place it in a complex developmental sequence.

On the evidence we have presented here, number conservation is complicated. It is not enough to collect judgments on conservation tasks and make a binary classification of conservers and nonconservers. Children's performance on related problems needs to be assessed, and multiple measures of performance (explanations, overt quantification, and response latencies) are needed. The classification system needs to be expanded to four categories: length responders (the classic nonconservers), quantifiers, transitional conservers, and conservers who can explain their judgments. Through such efforts, we should be rewarded with richer information about the relationships between number conservation and the quantifiers and operators that are its prerequisites.

ACKNOWLEDGMENTS

We wish to express our thanks to the staff and students of All Saints' Episcopal Day School, First Baptist Church Kindergarten Day Nursery, Teri Road Baptist Child Care Center, University Child and Family Laboratory, and Faith Presbyterian Church School in Austin, TX. We would like to thank Kevin Miller for his comments and Jean-Paul Fischer for his extensive editorial suggestions. An earlier version was presented at the Midwestern Psychological Association meeting, Chicago, May 8, 1986. A full report of the experiments is available in Tollefsrud-Anderson, Campbell, Starkey, and Cooper (1986).

8 Early Principles Aid Initial but Not Later Conceptions of Number

Rochel Gelman
Betty Meck
University of California, Los Angeles

Our account of number concepts shares with Piaget and Szeminska's (1941) the assumption that infants and children are actively involved in the construction of their knowledge. Unlike Piaget and Szeminska, we grant infants some skeletal domain-relevant structures, ones that help them search and use environments, assimilate and accommodate. We do not assume that such implicit knowledge is well-articulated. Early principles help children find and attend to what is relevant and therefore start them down the right developmental path. Although these structures are outline in form, they help keep together domain-relevant bits of material before their relationships to each other are understood.

By adopting a rational-constructivist view of cognitive development we do not expect immediate skill and perfection when children perform. Young children have a great deal to learn and their performance will vary as a function of experience, age, and setting. A key feature of our account of learning and development is that both mental representations *and* their class of potentially relevant inputs are defined with reference to abstract relations. We also offer an account of children's domain-specific errors that expands the list of error sources. In addition to the traditional ones—a lack of conceptual competence or random noise—errors can be attributed to limits on the ability to generate competent plans of action and/or to interpret correctly the question-answering routines used in a wide range of cognitive developmental assessments.

Although we presume there are some universal sets of skeletal principles underlying some early domains of knowledge acquisition, we do not believe that such principles are operative in all domains. Like Brown and Kane (1988), we maintain that our young have an advantage in their early learning if they already have some skeletal organizations with which to organize their memories of the inputs they are offered or find. Otherwise they have to acquire from scratch both the structure and knowledge of the data that is organized by the structure. We develop in this chapter this distinction between privileged and non-privileged acquisitions by comparing initial conceptions of number, ones that are rooted in counting and knowledge of specific numerical values, with ones that follow. For example, the idea that fractions are numbers is not consistent with the assumption that numbers are what one gets when one counts.

A BRIEF REVIEW OF OUR EARLY WORK

Gelman and Gallistel (1978) argued that investigations of the nature and acquisition of number concepts should take into account two kinds of related abilities: those that are used to generate representations of numerosities and those that are used to reason about numerosities. The achievement of mathematically meaningful representations of numerosity involves the ability to relate number-abstractors to number operations and relations (e.g., <, >, =, +, −).

The distinction between number-abstraction and number-reasoning abilities was first introduced in the context of an evaluation of Piaget's treatment of early numerical abilities. Gelman (1972) noted that young children's use of number operators is related to whether or not they achieve reliable and/or confident representations of the numerical values of the test displays. This led to the suggestion that Piaget and Szeminska (1941) were too quick to dismiss young children's counting as "rote" or non-meaningful, that is, not related to mathematical relations and operations. The idea was that studies of the relationship between numerical abstraction and numerical reasoning abilities might reveal the early presence of numerical concepts, ones that precede understanding of the mathematical operation of one-to-one correspondence.

The last twenty years has witnessed a veritable explosion of research on the ability of young children, including infants, to respond to numerically relevant inputs. Studies of numerical reasoning abilities have not been as ubiquitous. This turns out to be crucial to investigators' conclusions: (a) that infants' and very young children's responses to number are based on "subitizing," as opposed to counting; and (b) that early counting is not principled. We argue that both conclusions are premature. First, they typically are reached in the absence of a consideration of what is known or needs to be learned about these same children's knowledge of numerical operators and relations. Additionally, the source of performance variability is taken to follow from random noise or a lack of conceptual competence when there are reasons to assume other systematic factors are involved.

SUBITIZING OR COUNTING?

Some Reasons for Choosing 'Subitizing' as the Answer

Several recent papers have concluded that very young children do not use counting in a principled way to abstract a numerical representation of a collection (Fuson, 1988; Shipley & Shepperson, 1990a, 1990b; Siegler, in press; Wynn, 1990). Three kinds of reasons are given. First, the early numerical abstraction abilities of infants and toddlers is limited to small set sizes. For example, infants discriminate between the class of two and the class of three items: Their interest in looking at sets of three stimuli habituates and only recurs when the value of the set is changed to two, and sometimes, to four different kinds of items (Cooper, 1984; Starkey, Spelke & Gelman 1983, 1990; M. S. Strauss & Curtis, 1981). However, they do not discriminate reliably between sets representing larger values. Similarly, toddlers may answer "how many?" questions for sets of one, two, or three items with the correct cardinal term. Still they will fail to repeat the last tag they use when "counting" sets of this same size, and instead re-count when asked, "how many?," after they have already counted (Schaeffer et al., 1974; Wynn, 1990). Second, even older children who use count words to tag items in rather largish sets fail to distinguish between correct and incorrect counts and therefore, when asked, "how many?," after they count, will repeat the last tag whether or not it is the right answer (Fuson, Pergament, Lyon, & Hall, 1985). Third, the counting abilities revealed during the preschool years are variable and do not always generalize to novel settings (Baroody, 1984; Briars & Siegler, 1984; Wynn, 1990).

Ignoring for the moment that there are data that contradict or limit this pattern of findings (e.g., Gelman & Meck, 1986; K. F. Miller, Paredes, & Madole, 1989), we turn instead to the conclusions authors have reached about such patterns. First, very young children are said to respond to numerical displays by 'subitizing,' by reciting a number word they have learned to associate with a given visual pattern, much as they learn to associate common nouns with common objects. Second, the set size limit is taken as evidence that young children and infants do not count because they need not count: They can 'subitize' or 'perceive' number instead. Finally, the acquisition of a principled understanding of number is said to depend on learning—by imitation and association—bits and pieces of routines, procedures, habits, and so on, that look like components of counting. Only when children have acquired a sufficient amount of such knowledge, can they induce the principles of counting. A related assumption is that children's responses in experiments mimic the ways they learn to respond to environmental inputs in their everyday life. For example, it is assumed that children learn to repeat the last tag in a count sequence by rote because they see adults doing this and not because they realize that the last tag represents the cardinal value of the set (e.g., Fuson, 1988; Wynn, 1990).

Some Reasons for Rejecting 'Subitizing' as the Answer

It would be easier to evaluate the conclusion that very young children do not count and instead 'subitize' if there were a model of how the subitizing process maps a given numerosity into a unique symbol or state of the system that is estimating numerosity. Without this account, we have to assume that the idea is that "oneness" and "twoness" is processed much as is "cowness" and "treeness." Although we lack an account for the latter perceptual classification ability, there are some things that are surely true about it. Whatever the process, its description does not include an ordering principle such that cowness is always mastered before treeness, or such that the time needed to process *cow* will always be longer than the time needed to process *tree*, and so on. Therefore, the idea that the 'subitizing' process is like the processes that underlie object perception runs into difficulty explaining why oneness is always learned before twoness, twoness before threeness, and so on (Siegler, in press). Further, there is no account of why reaction times and error rates increase as a function of increases in set size, for both children (Chi & Klahr, 1975; K. F. Miller et al., 1989) and adults (Mandler & Shebo, 1982), no matter how small nor how much they are practiced. There are no comparable constraints on object pattern recognition. Additionally, there is no pattern perception model that is indifferent to all characteristics of the input it recognizes as exemplars of a kind. In order to recognize a cow, one must encounter cow-like stimuli, ones that look like cows, have cow parts, sound like cows, move like cows, and so forth. To some extent the size, color, age, posture, and so on can vary, but overall shape and kinds of parts cannot. In contrast, whatever the subitizing mechanism, there are no restrictions on the degree to which inputs can vary in terms of size, color, shape, or orientation. For geometric reasons there are some limits on the shapes that can be represented with a small number of distinct items but nevertheless there is always more than one way to arrange sets of at least two items. Likewise, there are many common arrangements that can be imposed on larger sized sets. Finally, if subitizing is a general perceptual process, why should judgments of numerosity, no matter how small the set, serve as inputs for numerical reasoning processes? What is there about the perception of cowness or treeness that would lead one to ponder the effect of adding or subtracting items or whether one display has more (or less) items in it? Yet, we know that babies as young as 12 months order different set sizes (Cooper, 1984) and take into account surreptitious changes in the number in an expected set (Sophian & Adams, 1987; Starkey, 1987).

Logic and Data May Not Be Enough to Change Theorists' Minds. In sum, neither logical considerations nor research findings support the conclusion that infants and very young children first use a standard pattern perception mechanism to discriminate between different numerosities. Even though there is much evidence against the belief that numerosity is simply perceived (K. F. Miller et al., 1989), the evidence does not seem to have much effect on adherents of the 'subitizing'-first hypothesis. Belief in the hypothesis does not even seem to be influenced by the ever-increasing body of research that shows that animals (another group of nonverbal subjects) are able to work with numerosities well beyond the presumed subitizing range (see Gallistel, 1989). Why is this?

We suspect that the persistent appeal to 'subitizing' as an account of early numerical discriminations follows from individual theorists' commitment to an empiricist account of learning. Within this frame of reference, primitive (i.e., initial) inputs are characterized in terms of sensory details (see Gelman, 1990, in press, for more on this point). To grant infants the ability to respond to the number of items on the basis of a skeletal set of counting principles (or some other number-based set of organizing principles) is to adopt a theory of the environment that is decidedly *not* empiricist in kind. However, this move does not, on its own, generate the corollary that infants come into the world with a full-blown, fully articulated, principled understanding of counting and number. What it does generate is an alternative way to construe the nature of learning and the definition of foundational inputs. Instead, relevant inputs for learning are defined with reference to structural considerations. This is a move that is not unique. It is quite common in accounts of perception (e.g., Marr, 1982) and fits well with the fact that children (young or not) contribute actively to their own cognitive development. Inputs that have potential to nourish children's nascent, skeletal knowledge are those that meet the structural definitions given by the domain (Gelman, 1990, 1991).

Some Reasons for Choosing Counting as the Answer

"A counting process is a process that maps from the numerosities of sets to symbols or states of the representing system" (Gallistel, 1990, p. 338). Such a mapping repeatedly assigns to a set of a given numerosity a unique symbol or state of the system that is an estimate of numerosity. Gelman and Gallistel (1978) coined the term *numeron* to refer to these representatives of numerosity in order to distinguish between what is being represented (numerosities) and what represents it (numerons). Numerons refer to the "inner marks" and support thinking about what Koehler (1950) referred to as "unnamed numbers." Gelman and Gallistel also provided an analysis of counting, by characterizing the formal properties of any process that we would want to call a counting process, conditions that have to be met by any competent plan for counting (Gelman & Greeno, 1989). The formal characteristics of counting processes are captured by three principles: the *one-to-one principle:* Each and every item in a set must be tagged with one and only one tag or numeron; the *stable-ordering principle:* Whatever the tags or numerons, they must be used in a repeatable ordered way; and the *cardinal principle:* The last numeron or tag used has the special status of representing the cardinal value of the set.

Notice that the one-to-one principle does not require that the items being counted be tagged in any particular order. It only requires that each item be assigned one and only one numeron. The absence of any such order requirement in the principles that define counting processes means that counting processes conform to what Gelman and Gallistel (1978) called the *order-irrelevance principle*. Notice also, that the three principles that define counting processes do not specify anything about the items in the set that is to be counted. They do not require that these items have any particular sensory characteristics, nor that they all have the same sensory characteristics, and so on. The absence of any such specification in the principles that define counting processes means that counting processes conform to what Gelman and Gallistel called the *abstraction principle*, which is that counting processes abstract from the items being counted only their distinguishability as distinct entities (Shipley & Shepperson, 1990a). It was perhaps an error to use the word *abstraction* here, for we did not mean to suggest that anything like a complex classification process is necessarily involved. Indeed, we noted that this condition of counting can be satisfied for objects if one merely has the ability to keep figure separate from ground.

We have already cited some of the findings that challenge our conclusion that skeletal sets of counting principles are available to help children attend to and learn about counting-relevant inputs. One especially problematic line of evidence is presented in Wynn (1990). In one of her tasks she asked 2- to 4-year-old children to count so as to indicate how many items were in a given display. Children younger than 3½ tended to simply recite as many count words as there were items and then to stop without repeating the last tag, the cardinal value of the set. When asked again "how many?," the young children re-counted. Fuson, Pergament, and Lyons (1985) reported a different problem in older preschoolers, sometimes as old as 5 years of age: They will repeat the last tag after both correct and incorrect counts.

Wynn also reported that her younger subjects (especially between 2 and 3½) solved her give-X task by grabbing two or three items no matter what the set size (other than one). Only "non-grabbers" counted out the number of items they were asked to give to a puppet. Wynn concluded that "grabbers" are especially disinclined to repeat the last tag after they count in response to a "how many?" question, doing so on only 26% of their correct count trials. In contrast, the group she called "counters" gave the cardinal value after 78% of their correct count trials. Because the tendency for children to give the cardinal answer after their count shifts abruptly at 3½ years—the same age that separates grabbers from counters—Wynn concluded that the younger children did not have a principled understanding of counting, that they do not understand that counting is related in a principled way to the cardinal value of the set.

THERE IS MORE THAN ONE SYSTEMATIC SOURCE OF ERROR

Granted, our theoretical life certainly would be simpler without findings like those just reviewed. But it is one thing to say these results deserve our attention and another to say our position has been dealt a mortal blow. First, those who reject our account have not dealt with the full range of the evidence we have developed. For example, neither Siegler (in press) nor Wynn (1990) considered Gelman and Cohen's (1988) finding that Down Syndrome children approach a novel counting task in qualitatively different ways than do normal preschool children. Similarly, our data from reasoning tasks (e.g., Gelman, 1977; Starkey & Gelman, 1982) are seldom mentioned. Finally, our efforts to explain failures in terms of systematic non-arithmetic sources of error are either ignored or treated as a claim on our part that young children do not have the vocabulary in question. Our position regarding the kinds of non-numerical knowledge needed in particular task settings goes well beyond this.

Siegler (in press) is correct in his observation that investigators would do well to replicate their critics' results and then go on to variations of the original design. Although Gelman et al. (1986) did not do this in their response to Briars and Siegler (1984), they did in their response to Baroody (1984). Furthermore, in ongoing research we also follow the strategy of replicating problematic results before systematically manipulating potential sources of variation other than a lack of the domain-specific *conceptual competence*, the implicit knowledge of the counting principles. Other sources include (a) inadequate *procedural competence*, which supports the generation of plans of action; and (b) limited *interpretative competence*, that constellation of abilities that allows one to understand how to deal with tests, to take settings into account, and so on.

Analyses like those in Gelman and Meck (1986), Gelman and Greeno (1989), and Greeno et al. (1984) led us to ask both whether we could adduce evidence of conceptual competence in children younger than 3 years and show that limits on interpretative or procedural competence interfere with young children's ability to reveal such early conceptual competence. It is reasonable to suppose that children under 3 are still developing skill at inhibiting prepotent response tendencies and matching the correct response class to a task's requirements—the kinds of skills that contribute to procedural competence. Our studies of the potential role of procedural competence in the "make x" (or "get x") task are not far enough along to merit discussion here. Similarly, because interpretative competence is underdeveloped at this age (Shatz, 1983; Siegal, 1991), it, too, could have affected the performances already summarized. We turn now to this possibility.

New Findings with Adult Counters: The Gelman, Kremer, and Macario Studies

The more we thought about the matter, the more we wondered why anyone would repeat the last tag after counting a single array in response to a "how many?" question. So, with Kathleen Kremer and Jason Macario, we asked adults (UCLA undergraduates enrolled in an introductory psychology course) to answer this question. In our first study, we told each student that we wanted to see how adults do on tasks that we have used with children. As we dropped 18 blocks in front of them (resulting in haphazardly arranged displays), we asked, "How many blocks are here?" We told them to think out loud while figuring out how many there were, but we did not tell them to count. All 13 subjects counted—sometimes using grouping strategies, such as counting by twos. Only one repeated the last tag spontaneously, and this was the only individual who made a counting mistake.

Our subjects' failure to tell us the cardinal value after they counted led to a second version of the procedure. Now, we repeated the "how many?" question if subjects did not repeat their last counting tag after they counted. The follow-up question was presented with a flat to falling pitch contour, so as to avoid a challenging tone. Of the 10 students in this variant of the task, 4 simply repeated their last count tag when asked, "how many?" again. The remaining 6 behaved as though they thought the question was odd. One person said, "Huh?," another laughed nervously, another thought we meant that she got the wrong answer and therefore re-counted, and still another said, "Eighteen, I hope." The remaining 2 subjects registered surprise and/or raised their voices.

Might adults treat "how many?" questions differently when they are with young children? Cohen's (1987) observations of adult-child pairs who visited the "How many?" Box at the Please Touch Museum in Philadelphia suggest they do not. First, only one third of adults actually read the "How many?" sign to their charges who were all too young to read by themselves. Second, those who did almost always accepted counting strings as answers. They neither offered the cardinal value themselves nor asked questions like, "And how many is that?" Gelman and Massey (1988) presented a related report.

Recall the associationist assumption that children learn to repeat the last tag because they are reinforced for imitating the input they are offered. The preceding results open the possibility that children do not encounter the assumed input. Instead, adults seem to assume that they should *not* repeat the last tag when they are shown a set and asked, "How many?" If number is taken as the shared topic of conversation, there is no need to repeat what is obvious to both speaker and listener, that is, that the last tag answers the "how many?" question. Repeating the target question violates Grice's conversational maxim of quantity: Do not say more than necessary.

Given the foregoing, we conclude that the seemingly straightforward "how many?" task affords a potential confound of preschoolers' ability to repair violations of conversational rules with assessments of their cognitive abilities (Donaldson, 1978; Siegal, 1991). This is especially so if a child responds to a "how many?" question by counting and is then asked this question again. Children learn relatively late that it is permissible to repeat or tell what is already known by the speaker in test settings (Heath, 1983). Therefore, data from the "how many?" task are as consistent with a developmental account that emphasizes acquisition of the knowledge that violations in conversational rules are allowed in test settings as is an account that denies young children counting competence. If so, we need to find settings where repeated requests to count and answer "how many?" questions are less likely to be conversationally anomalous before making a choice between these alternative accounts. The "Magic" paradigm turns out to offer such a setting.

Bullock and Gelman (1977) Reconsidered

Bullock and Gelman (1977) used the Magic paradigm, a two-phase game, to determine whether 2½-, 3-, 4-, and 5-year-olds could reason with and about numerical relations. The children were tested individually. In Phase 1 the experimenter hid one item on a plate under one can, and two items on another plate under a second can. *Without mentioning the numerical values*, the experimenter proclaimed one display the *winner*. Half the children saw the experimenter point to the 1-plate and call it the the *winner* (the less condition); half were told the 2-plate was the *winner* (the more condition). The displays were then covered and shuffled. The child had to guess which one had the winner, look under the chosen can, and then decide whether they were correct. If the child chose the "wrong" plate but correctly labeled it the *loser*, she was allowed to find the winner under the other can. Errors of identification led to the start of a new trial. Phase 1 continued for 10 or 11 trials, or until a child reached criterion. Phase 2 started when the experimenter surreptitiously added two items to each display (something that was easy to do with young children because they liked looking at the prizes they collected for correct Phase 1 identifications). During Phase 2, the children had to decide which of the uncovered new values (3 or 4) was a *winner*. Those who were initially rewarded for finding 1 now had to designate 3 the *winner;* those initially rewarded for 2 had to choose 4 to be scored as correct.

In Bullock and Gelman's first experiment, 3-, 4-, and 5-year-olds succeeded in what we henceforth call the *regular condition*. In contrast, the 2½-year-olds in the same condition did not choose correctly on the basis of a common numerical relationship between Phase 1 and Phase 2.

Bullock and Gelman conducted a second experiment, henceforth called the *control condition*, to test the hypothesis that the youngest children in the regular condition simply did not think to transfer the knowledge they acquired in Phase I to deal with Phase 2. Two variants of the follow-up experiment provided hints to do this. In one, the initial displays were left in place with their covers on; in the other, the initial displays were left in place uncovered. Then, in Phase 2 of the control condition, the experimenter introduced a new game "like the one we just played" and put two new covered displays on the table. Whether the Phase I displays were left covered or uncovered had no effect. In both variants of the control condition, a reliable number of 2½-year-olds chose relationally, selecting in Phase 2 either the 4-item (more) or 3-item (less) as a function of their Phase I reinforcement.

In the Magic paradigm we sporadically asked the children why a given display won or lost during Phase I. After the children answered the Phase 2 questions for the altered display(s), we interviewed them to determine whether they knew how many items were present during both Phase 1 and 2, how many were added or removed, and so on. Because these questions were asked for different values that were either physically present or to-be-remembered, this is a setting where they should have found it acceptable for us to repeat "how many?" questions. Because the children could compare different set sizes, they had reason to count more than once. Further, by asking how Phase I and Phase 2 displays differed, we also gave them opportunities to reason about the numerosities represented. Given these considerations we returned to the Bullock and Gelman transcripts to see if they contained evidence that bears on the present discussion.

A Reanalysis of Bullock and Gelman's (1977) Youngest Subjects. The data under consideration here come from the transcripts of the twelve 2-year-olds in each of the control and regular conditions as well as the 12 (of 24) youngest children (those less than 42 mos) in the 3-year-old regular condition. We included the latter children because they, too, are within the age range in which, according to others, children lack the verbal cardinal principle. Bullock and Gelman focused on: (a) children's answers to the beginning Phase 2 "Which is the winner?" question; (b) whether the children knew how many items had been and were present on the displays; and (c) how much the children knew about the relevant transformations. Although Bullock and Gelman presented some data on children's tendencies to talk about number, they did not determine whether the children applied the verbal cardinal principle. To do this, we considered all Phase 2 talk about number that occurred following the surprise trial, keeping track of whether it was spontaneous or elicited, and what the experimenter said before and after each utterance that included number talk.

Wynn's (1990) findings and conclusions led us to focus on whether children's use of number in Bullock and Gelman's study reflected an ability to count and say the cardinal value for set sizes of 3 or smaller. Wynn acknowledged that young children will count small sets when asked to do so; her claim was that they will not count and offer the cardinal value of such sets when asked about cardinal number, as is the case for the "how many?" question. Therefore, we analyzed the children's talk about number in two ways. The first made use of all number talk, including any that occurred in response to the experimenter's request to count. For the second, all number talk that followed a request to count was excluded. The different data bases are clarified by a consideration of samples of the Bullock and Gelman transcripts given further on. Subject #3, for instance, was asked to count about halfway through the excerpt, and Subject #9 was asked to count near the end of her interview. Their subsequent answers formed part of the data for our first analysis of whether the young children in Bullock and Gelman's study applied the cardinal principle. Such answers were excluded from the data base used for the second analysis. Therefore, for our first analysis, we scored a child as having given both a cardinal and counting response who (a) did so spontaneously, (b) offered two kinds of answers to the same questions (be it "How many?" or "Can you count . . . ?"), or (c) when asked to both count and answer the "how many?" question for the set size, gave both a counting and a cardinal response. When we redid the analyses, such count or cardinal answers to our request that a child count were ignored.

Wynn concluded that children do not mix 'subitizing' and counting strategies once they apply the verbal cardinal principle, that is, once they show evidence of understanding that counting generates the cardinal value of a set. For us, this meant that we could conclude that the young children in the Magic experiment revealed a principled understanding of the verbal cardinal principle only if they both counted *and* gave cardinal values for small sets. Children who simply counted, simply stated a numerical value, or failed both to count and to state a cardinal value for small sets, provided negative or ambiguous evidence for our position. Finally, Wynn reported that there were 4 (of 22) children who counted and applied the cardinal principle in her main study. Therefore, in order to affirm our hypothesis, the proportion of children in Bullock and Gelman's data scored as having counted and applied the verbal cardinal principle had to be reliably larger than this.

Bullock and Gelman reported that their youngest subjects were disinclined to talk about number during Phase 1. The transcripts showed that they were nowhere near as reticent in Phase 2, especially in the control condition. The youngest children in the regular condition made more number-use errors; they also seemed more tongue-tied at the start of Phase 2, possibly due to the surprise generated by the violation of expectancies. Still, by the end of the interview, every child, whether in the regular or control condition, used some number words correctly, even though they were never explicitly corrected for earlier errors. (This tendency to improve without explicit feedback fits our model. Skeletal principles give young children a way to first identify potential number-relevant environments and then engage in on-line, but not conscious, monitoring and self-corrections. See Gelman & Cohen, 1988.) The kind of conversation that occurred in this setting is illustrated in the following sample of protocols. (Subjects' comments are in italics. Their spontaneous Table 1 count code ends a protocol.)

S#9 (34 months). [Regular-2s Condition. Reinforced in Phase 1 for 1 (less). S never said why correctly identified displays were winners or losers.]

Phase 2, Trial 1: S selected and uncovered the 4-item display. "Well, is that the winner?" *Why did you put three in there?* "How many are on there?" *1-2-3-4-5* (5 emphasized). While E goes on talking, S continues: *1, it should be . . . ; three. Oh, 1-2-3-4, that's four.* (E continues to ask which wins, which is the best winner, and so on. S does not use numbers throughout the next quarter of the interview.) When S designates the 3-plate the winner and says: *Best,* the experimenter asks, "Why?" *Three?* "Huh?" *This be two, or one (4-plate). Two, two* (3-plate). (S possibly wants to pretend that the new displays are the old 1-item and 2-item displays). "How many are on this (4) plate?" *1-2-3-4.* "How many are on this (3) plate? Can you count them?" *1-2-3.* "So which one is the winner?" *This one?* (pointing to the 3-plate again). "Why?" S leaves the table to play with prizes. . . . S returns and resumes counting correctly. S also counts a 2-item display, *1-2* in response to, "How many?" [Card +, Count +]

S#3 (34 months). [Control-2s Condition. Reinforced in Phase 1 for 2 (more). Phase 1 plates left covered on the table when 3- and 4-item displays (with covers) are put on the table to start Phase 2.]

Phase 1, Trial 4: S chooses a covered plate and correctly identifies it once uncovered. "Why is that the winner?" No answer. Trial 5: Wrong guess of which can, but correct at saying it is not the winner once can is lifted. "Why is that the loser?" *I don't know.* (Pattern repeats for Trials 6 and 8, the other Phase 1 probe trials.)

Phase 2: "I've got another game in my bag; want to see it? *Yes.* "It's kind of like another game. What do you think, J, which one's the winner?" J guesses, picking one and then the other . . . "How many are on this plate. Can you count them?" *Oh.* "Can you count them? How many are here? (4)?" *Four.* "How many ducks are here (3-plate)?" *Five.* "Can you count these (4-plate) with your fingers?" *1-2-3-4.* "That's terrific, J . . . Do you remember how many used to be the winner?" *This.* (Points to a Phase 2 plate). "Before, how many used to be the winner?" *1-2-3-4-5.* . . . "Let's look at these plates. Which one is the winner?" *Two.* "How many on here?" *One.* "Now which is the winner over here" (Phase 2 plates). *This is the - that's the loser* (3-plate) *and this is the winner* (4-plate). "Why is that the winner?" *You win.* (J starts reciting count string.) [Card +, Count −]

S#11(35 months). [Regular-3s Condition. Reinforced in Phase 1 for 2 (more).]

Phase 1, Trial 4: S correctly selects and identifies the winner. "Why is that the winner?" *Cause it has a bird.* Trial 5: S selects the winner, correctly identifies it, and says: *The bird, the bird and then there was a bird.* "Why is that the winner?" *Cause it has two birds.* Trial 6: S uncovers the loser and says so. "Why is that the loser?" *Cause it is.* Trial 7: Correct guess and identification. "Why is that the winner?" *The winner 'cause it has two birds.* Trial 8: "Why is that the loser?" *'Cause it has one bird.*

Phase 2, Trial 1: S uncovers a 3-plate. "Is that the winner?" *Yes.* "How many are on there?" *1-2-3.* "What's under this can?" *1 bird, 2 birds, 3 birds, 4 birds.* "Well, which one is the winner?" *This one's the winner; it has 1-2-3; there's three birds.* S continues, making both new plates into 2-item plates by removing the right number of items. Finally, she returns to making a 3-plate and 4-plate and shows the latter while saying it wins. E continues, "How many does the winner have?" *1-2-3-4.* Again, S makes two 2-plates. E asks, "How many does the winner have?" *1-2.* [Card +, Count +]

Table 8.1 summarizes, subject by subject, the outcome of our analyses of both probed and spontaneous number use. For each child the table includes his or her age (in mos) and comments about the kinds and/or locus of the errors made and whether there were enough trials to warrant an attribution of competence. We use the term *Card* (cardinalize) where others would use *Subitize,* that is, when the child used a single count word to refer—either in response to a question or spontaneously—to the numerosity of a set. The terms Card and Count are modified + or − to indicate if a child did or did not Card or Count.

Regardless of whether we included answers following requests to count, the majority of children were scored as Card+, Count+, having given evidence of using the verbal cardinal principle. The number of observed Card+, Count+ children is greater than expected, whichever analysis is considered. Using Wynn's data as a baseline, we compared our data with hers and obtained (X_1^2 = 7.09 and 5.66, $p < .02$). Inspection of Table 8.1 makes clear that the tendency to simply state the cardinal value without counting, was not a function of age. If anything, it was differences between the conditions that determined the level of numerical talk during Phase 2.

In sum, our reanalysis of the Bullock and Gelman data provides further evidence that even young children try to relate their efforts to learn to count to their implicit understanding of counting principles. There is no question that early knowledge about the conventions of counting is fragile. It would be surprising if this were not so. Count lists vary across language groups and some are easier to master than others (K. F. Miller & Stigler, 1987). Even within the United States different cultural groups have different ways of interpreting conversational postulates and their violations (Heath, 1983). Such matters of convention for a given culture are exactly the sorts of things children have to learn (Gelman & Greeno, 1989). By granting to our young skeletal knowledge—

TABLE 8.1
Reanalyses of the Transcripts from the Bullock and Gelman (1980) Magic Experiments to Assess Whether Children Younger than 3½ Applied the Cardinal Principle

Condition & Age of Each S	Relevant Features of Number Use in Phase 2*	Attribution of Cardinal Principle: Including Data from Requests to Count — Summary of Counting Ability	Including Data from Requests to Count — Grant Cardinal Principle?	Excluding Data from Requests to Count — Summary of Counting Ability	Excluding Data from Requests to Count — Grant Cardinal Principle?
Control Condition, 2-year-olds					
29 mos	Many count errors on large numbers of prizes; hard to code	Card^{+}, Count^{-}**	No	Card^{+}, Count^{+}	No
30 mos	—	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{-}	No
32 mos	No card responses; counts correct only on $x < 4$	Card^{-}, Count^{+}	No	Card^{-}, Count^{+}	No
34 mos	One card error; easy to code	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{-}	No
34 mos	Card values wrong; used like adjective slot fillers	Card?, Count^{-}	No	Card?, Count^{-}	No
34 mos	Both Card and Count correct on 1-3; count errors on 4	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
35 mos	Perfect counts on last half trials; errors to start	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
35 mos	—	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
35 mos	Never Cards	Card^{-}, Count^{+}	No	Card^{-}, Count^{+}	No
35 mos	—	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
35 mos	Never Cards; not clear can Count	Card^{-}, Count^{-}	No	Card^{-}, Count^{-}	No
35 mos	—	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
Regular Condition, 2-year-olds					
29 mos	One early counting error	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
30 mos	—	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
30 mos	One count error (of 3) on 4	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
32 mos	Not enough relevant data	Card^{-}, Count^{-}	No	Card^{+}, Count^{+}	Yes
33 mos	Only 1 count error on 4	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{-}	No
33 mos	Errors; best guess is Card only	Card^{+}, Count^{-}	No	Card^{+}, Count^{-}	No
33 mos	One count error (of 4) on 4	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
34 mos	Wrong Card value; count errors on large xs	Card^{-}, Count^{-}	No	Card?, Count^{+}	No
34 mos	Count errors in first half	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
34 mos	Card only	Card^{+}, Count?	No	Card^{+}, Count?	No
35 mos	—	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
35 mos	Not enough counting data	Card^{+}, Count?	No	Card^{-}, Count^{-}	No
Regular Condition, 3-year-olds					
36 mos	Errorful counts; Cards "2" for 4	Card^{-}, Count^{-}	No	Card^{-}, Count^{-}	No
38 mos	Errors on 5; fine for $x < 5$	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
38 mos	Count errors on 3	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
38 mos	—	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
38 mos	Card values used like adjectives; not enough counting	Card^{-}, Count^{-}	No	Card^{-}, Count^{+}	No
39 mos	Not enough counting	Card^{+}, Count^{-}	No	Card^{+}, Count^{-}	No
39 mos	—	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{-}	No
39 mos	Early count error of 5 prizes	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
39 mos	Easy to code; count errors on large x	Card^{+}, Count^{-}	No	Card^{+}, Count^{+}	Yes
40 mos	—	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
40 mos	—	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes
40 mos	Count error on 5	Card^{+}, Count^{+}	Yes	Card^{+}, Count^{+}	Yes

*If coding was straightforward, this column was left blank. Code based on post-surprise, Phase 2 talk.
**Card$^{+/-}$ = Applies/Does not apply cardinal principle; Count$^{+/-}$ = Knows how/does not know how to count in range tested; Count^{-} = Does not know how to count in range tested. See text for further details.

some principles of counting and arithmetic reasoning—we do not take away their need to learn. Rather, we give them a way to help themselves find what their culture offers their active tendency to use the knowledge they have, to assimilate and then to accommodate.

INITIAL PRINCIPLES OF COUNTING AND REASONING CAN INTERFERE WITH, AS WELL AS ENCOURAGE, LEARNING ABOUT NUMBER

So far we have focused on the idea that the acquisition of number concepts benefits from skeletal counting and reasoning principles. Like all constructivist theorists of knowledge acquisition, we assume that children will sometimes achieve the "wrong" interpretation of the data we offer them because they apply different knowledge structures than we do. The case of fractions serves to make this point, while showing in another way that young children have a potent tendency to apply their counting principles.

Do Young Children Know That Counting Numbers are not the Only Kind of Numbers?

It is not just young children who use counting algorithms to solve arithmetic problems; so do unschooled adults. Schooled or not, individuals have strong tendencies to decompose natural numbers into manageable or known components, to count when adding or subtracting, and to use repeated addition (or subtraction) to solve multiplication (or division) problems. Children use these proto-multiplication and -division solutions before and after they learn more standard multiplication and division algorithms in school; so do unschooled adults and children in a variety of settings and cultures (T. N. Carraher, D. W. Carraher, & Schliemann, 1985; Saxe, 1988; Starkey & Gelman, 1982).

Some of the algorithms invented by older children and unschooled adults are more complex than those used by preschoolers. Still, as Resnick (1986) noted in her analysis of these, all use principles of counting and addition (or subtraction) with the positive integers. Her subject, Pitt (7 years, 7 months) illustrates the point, solving 2×3 "*Two threes . . . one three is three, one more equals six.*" Similarly, schooled or not, individuals in Africa and Latin America use a combination of number decomposition moves and repeated additions (or subtractions) to solve investigators' multiplication problems. Young children are less able to work with large numbers than older children or adults, but beyond this it is hard to distinguish their invented, out-of-school, solutions from older children's and adults' (T. N. Carraher et al., 1985; Saxe, 1988).

We believe that the widespread invention of algorithms that are based on counting and/or repeated addition (subtraction) algorithms provides further support for our conclusion that a universal set of implicit principles governs the

acquisition of initial mathematical concepts (Gelman, 1982b). Still, over the centuries, there have been changes in the definition of what is or is not a number, changes that are well-described in the history of mathematics. Much of mathematics involves operations and entities other than counting and the addition and subtraction of the counting numbers (or the positive integers). Might it be that when it comes time to transcend the early knowledge built on the counting principles these principles will hinder progress almost as much as they first promoted it?

We take the data on unschooled mathematical abilities as evidence that numbers are first thought to be "what one gets when one counts things." If so, common classroom inputs for lessons about fractions may be misinterpreted by young pupils. One cannot count things to get the answer to, "Which is more, ½ or ¼? 1.5 or 1.0?" Number lines are not simple representations of whole numbers, yet young children might think they are. Similarly, young children might know that one can take apart a circle and a rectangle and get two "halves" on both occasions and still not appreciate why each of these halves can be represented with the written expression ½. The interpretation of the latter might be assimilated to the idea that such marks on paper must be about whole numbers.

In the absence of implicit principles for dealing with fractions, young learners might overgeneralize their counting principles and produce a distorted assimilation of the instructional data on fractions. For example, if they do assimilate their understanding of the language of fractions to the implicit system of mathematical principles available to them, they might "read" fractions, or non-integer numerals, as if these, too, were representations for the counting numbers. For example, they might choose ¼ as more than ½; or, when asked to place 1½ circles on a number line, they might decide they have two things and place the stimuli at the position for 2 on the line. Such speculations have proven to be right.

Gelman, Cohen and Hartnett (1989) found that children in kindergarten, and first, and second grades repeatedly treated fraction tasks as if they were new occasions for counting. For example, 85% of 40 children misread the written expressions ½ and ¼. Many read the fractions as two integers, "1, 2"; "1 and 2". Others read the division symbol as *and*, *plus*, or *line*. Finally, some children behaved as if they had been given an addition problem; when asked to read ½ and ¼, they answered, "three" and "five," or even "1 plus 2, 3, 5, and 1 plus 4"! Additionally, the children in the study had a reliable bias to select the reciprocals with the larger denominators as "more" when they were shown ¼ versus ½ and $1/56$ versus $1/76$.

These and other results in the study make it clear that most of the children treated the test materials as novel occasions for counting and reasoning about whole numbers. Our inputs about fractions offered them further cases to which to assimilate their view that numbers are what one gets when one counts and adds counting numbers. Does this mean that learning about fractions as numbers will be difficult? There is reason to conclude just this.

Formally, fractions are numbers generated by the division of two numerby counting the number of things or parts of things. The formal ...iples for counting and addition do not include those that define the relations between multiplication and division: The distributive law is not a law of addition but a higher-order one. Therefore it is not clear how mathematically meaningful learning about fractions could be built from what is already known by young children. In other words, learning about fractions *should* be difficult. More generally, learning about fractions seems to depend on being able to use, in mathematically meaningful ways, conventional mathematical terms, tools, numerical symbols, and the notational systems for writing fractions. Thus, the mastery of the idea that fractions are indeed numbers requires both the learning of principles that go beyond those implicit, foundational ones available to the young child just as correct mathematical language does. (See also Hiebert & Wearne, 1986.)

Gelman and Gallistel (1978) reminded their readers that the formal description of arithmetic is a product of late 19th- and early 20th-century mathematics. Its emergence was accompanied by a profound shift in mathematicians' views about the relation between the laws of arithmetic and the definition of what constitutes a number. Whereas the 19th-century mathematician Kronecker assumed that the natural numbers were given and all else derived, there is a sense in which almost the opposite is true now. No matter how bizarre it might seem from the psychological point of view, in modern mathematics, a number is any abstract entity that can be shown to behave in accordance with the laws of arithmetic. Whatever intellectual discomfort negative numbers generate—one cannot count $-x$ real-world things— they are an essential part of modern formal arithmetic. Similarly, no matter how articulately a child denies numberhood status to fractions, they are numbers in modern, formal arithmetic. The property of closure with respect to division would not apply otherwise: It will not do to discard the remainder in a division problem simply because it is not a whole number.

If learning to go beyond the initial conception of number requires mastering some of the language and syntax of mathematics, then we are faced with a special challenge. The idea that there are noncounting numbers for purely formal reasons is historically young. Therefore, it is most unlikely that learning about concepts of number that are so characterized benefits from existing skeletal structures. If this is, indeed, the case, children not only have to develop new mathematical structures, but must learn how their entities behave mathematically as well (Gelman & Greeno, 1989).

In our introductory remarks, we took care to state that we do not believe that all learning about mathematics benefits from or is based on early skeletal principles. It only makes sense to grant learners the ability to actively find and learn about relevant inputs on their own when they have available structures with which to do so. In the absence of structure relevant to the concept(s) in question, learning is known to be exceedingly hard, even with the aid of structured inputs. Because much of mathematics involves operations and corresponding entities

other than those relating to counting and addition with the counting numbers, mastery of the domains of mathematics has to involve development of novel structures. It is even possible that when this happens initial conceptions of number change in a deep way. Future work on the development of number concepts must focus on these matters. Doing so would be very much in the tradition of Piaget.

ACKNOWLEDGMENTS

Preparation of this chapter was supported by NSF grants BNS 85-19575 and BNS-8916220 as well as the UCLA Dean's fund to Rochel Gelman. We thank C. R. Gallistel, Kathleen Kremer, and Jason Macario for their thoughtful insights about the nature of the issues discussed here. Special thanks are due Jason Macario for his careful comments at every stage of manuscript preparation. We are especially grateful to our editors and publishers for providing us the oppertunity to note some matters of fact that bear on some of the ways that the Baroody chapter has been updated from the French to include considerable commentary on our chapter.

Addendum

Rochel Gelman
Betty Meck
University of California, Los Angeles

Barody's Chapter 5 in this volume incorporates a variety of responses to our chapter. Although we edited occasionally to clarify what we wrote (see following), our chapter parallels the French version. What follows is neither an inclusive list of pertinent points nor a rebuttal to Baroody. We assume that readers will want to consult original sources on their own. Here, we list some especially salient matters of fact:

1. Contrary to the conjecture in footnote 8 (p. 108), the reversals in the count lists used in the stable-order error-detection conditions for set sizes 5, 7, and 20 were *not* restricted to the first five count words in the Gelman and Meck (1983).

2. Regarding footnote 13 (p. 117), it actually was the normal preschool children, not the school-aged, Down Syndrome (DS) children, who were in the harder condition in Gelman and Cohen (1988). For half of the preschoolers, the position of the item they were to "make the X" varied. In fact, the position of the targeted item for a DS subject varied between but not within subjects. Further, the DS children received intensive training in counting and therefore, if Baroody's account is correct, they should have learned more. Nevertheless, preschoolers outperformed 8 of the 10 DS children on a variety of criteria. For example, they were more inclined to self-correct and/or *self-initiate* a new attempt to solve the novel task. They also were able to benefit from indirect hints, whereas DS children even failed after watching a sample solution.

3. Figure 5.1 (p. 120) is based on a preliminary report to Baroody of the final work published in Gelman and Meck (1986).

4. Regarding footnote 10 (p. 112), children did see *E* point as she counted during the trick trial. Of course, she did not point conspicuously, otherwise she would not have met the design conditions of the trial. Children gave different kinds of explanations on trick trials as opposed to one-one error detection trials, an outcome that mitigates against the suggestion that we have a Clever Hans effect (footnote 11, p. 113).

5. Given Baroody's concerns (p. 113) about S#3, we did a full review of our summary data sheets and found two copying errors: (a) S#3 should have been entered as a 34-mo-old, and (b) the first child (a 29-mo-old) did not meet the spontaneous Card+ and Count+ criteria. Therefore, the rightmost column of Row 1 in Table 8.1 (pp. 184–185) is changed to *No*. The requisite recalculated chi-square value appears on page 183 and is still significant. This analysis is not yoked to the arithmetic reasoning analysis that was published in the original paper. During Phase 2 of the magic experiment, the nonreinforced surprise trial(s) was followed with questions and discussion. The analyses in Table 8.1 are based *only* on the way count words were used after the surprise trial in Phase 2. All counts from this point—be these of each test display, a combination of the items on both plates, or the prizes a child accumulated during Phase 1—were scored to arrive at the Count and Cardination data shown in Table 8.1.

6. Our adult counting study (pp. 178–179) makes the hypothesis that young children learn a cardinal rule on the basis of opportunities to imitate and be reinforced by their elders less plausible. We find that adults, like young children, do *not* repeat the last tag when answering "How many?" questions. Indeed, they get uncomfortable in a follow-up study that asks the How many? question again, presumably because conversational rules are violated. See Siegal (1991) for further cases of tasks that confound the assessment of interpretative and conceptual competence and excellent discussion of the questions raised by Baroody on pages 123–124.

7. Baroody writes that "learning plays a role in many behaviors that were once thought to be innate" (footnote 18, p. 125) and offers several examples, including bird song. This is true, but it does not follow that terms like *learned* and *innate* are treated as opposites by all modern ethologists and animal learning theorists who study these kinds of complex acts. For example, in his chapter, "The Instinct to Learn," Marler (1991) developed a detailed account of how an innate template helps the immature male white-crowned sparrow attend to and store examples of his adult song, well before he can sing or even enters the extended periods of practice and learning that must follow before he will sing a mature adult song. See Gallistel, Brown, Carey, Gelman, and Keil (1991) for further examples of privileged learnings.

Author Index

A

B