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Toward an Understanding-Based Theory of Mathematics Learning and Instruction, or, In Praise of Lampert on Teaching Multiplication

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Dr. Lampert has written a seminal article on the psychology of mathematical learning and instruction. She provides a clear treatment of the emerging constructionist theory of mathematics learning and, hence, an alternative to theories that appeal to an associationist framework to explain the acquisition of arithmetic knowledge (e.g., Siegler & Schrager, 1984). Dr. Lampert's goal is to get pupils to construct an understanding of the mathematical principles that constrain the class of possible multiplication algorithms. She argues that to do this one must integrate both the computational and the conceptual – or both the procedural and the principled – approaches to the design of mathematics curricula. Her proposal is that curricula that satisfy this criterion will lead to the kind of mathematical understanding that underlies an ability to generate novel solutions to new problems.

In defense of her thesis, Lampert offers a case study of how students learn when exposed to a curriculum designed within this framework. We are presented with clear examples of children inventing new algorithms, sometimes getting it right and sometimes making systematic errors reflecting the presence of at least *some* implicit understanding. An especially exciting feature of Lampert's work is her ability to connect her theortical efforts to a significant problem in pedagogy in an inventive, insightful manner, all the time keeping track of her goal to teach multiplication algorithms in a way that brings children to understand the mathematics involved. Would that all children learning to multiply had so gifted a thinker and teacher of mathematics.

Lampert opens her article with a sobering reminder: The current debate on whether we should teach mathematical concepts (now often referred to as

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conceptual competence) or algorithms (*procedural or computational competence*) has a very, very long history—much longer, I fear, than many of us know (however, see Hiebert, 1986). Rather than continue in the spirit of this debate, Lampert refuses to choose sides; indeed, she develops the position that we should face the fact that we have no choice but to teach children both conceptual and procedural knowledge. This is possible, she maintains, if we choose our materials with the goal of being able to illustrate the mathematical meaning embedded in algorithms. To accomplish this, Lampert designed a curriculum to teach children that they could generate novel instances of procedures for doing multiplication if they could master the principles that constrain the class of possible algorithms. Lampert's article amounts to an existence proof because it provides compelling evidence that her pupils did achieve the level of mathematical understanding needed to choose possible new procedures.

Lampert came to her teaching plan after a careful consideration of certain developments in the cognitive sciences, the ones I believe are contributing to the development of a successful theory of mathematical and scientific pedagogy. The critical ingredients of this evolving theory (for examples in these domains, see Gelman & Greeno, in press; Greeno, Riley, & Gelman, 1984; Leinhardt, in press; Resnick & Omanson, in press) are the following:

1. Children are active participants in the construction of their own knowledge.
2. Materials should be presented so as to allow children to take advantage of what they already know about the principles of counting and arithmetic, the nature of different symbol systems, number facts, and related algorithms.
3. Procedures or algorithms take on meaning to the extent that they are recognized as the outputs of plans that are constrained by knowledge of the principles they serve.
4. The goal of mathematics (and I would add scientific) instruction is to expand principled understanding of the domain (i.e., to teach children the mathematics underlying the algorithms they use to get answers to problems).
5. One way to develop principled understanding might be to teach children to recognize that different algorithms can instantiate the same principles.

A focus on different algorithmic instantiations of a set of principles helps teach children that procedures that seem very different on the surface can share the same mathematical underpinnings and, hence, root meanings. After all, it is because different procedures share a core set of principles that we can say they serve the same function or represent an equivalence class with

respect to the same goal, in the present case, how to generate the correct answer to a multiplication problem.

Some will say that it is all well and good to have such lofty criteria, but everyone knows they cannot be met. If so, the Lampert article would not exist. Indeed, I take Lampert's success with her pupils as strong evidence for the class of theories she endorses. The ability to generate novel solutions to problems within a domain often counts as clear evidence of at least implicit understanding. Just as we say that speakers of a language have implicit understanding of the rules therein because they can generate novel utterances, so we should say that pupils who generate novel routes to their answers have at least implicit understanding of the mathematical principles involved.

There are other criteria one could use to assess understanding within a domain. One that is often used in psycholinguistics is the ability to recognize and accept examples of novel utterances that are in the speaker's language and to reject strings of words that are not. Lampert builds this criterion of novelty into her curriculum because she expects children to respond to novel cases introduced either by her or by others in the class. One's ability to recognize that someone else has used an acceptable case often can be exceedingly reinforcing—I still recall an entire third grade cheering one of their own when he proposed that "purple elephants" constituted an example of an empty class. I suspect that children in Lampert's class likewise experienced positive affect when they saw that the solutions of other children, as well their own, were acceptable. This is exactly the kind of feedback Piagetians tell us is part and parcel of using our knowledge structures to identify and assimilate examples that can feed our construction of structures.[1]

All of this talk about teaching concepts, principles, structures, and so forth may lead some readers to ask: Are Lampert and/or Gelman recommending that teachers stand up in front of a class and say, "Now children, today we will learn the distributive law; but first let us review the law regarding the composition of numbers"? *Absolutely not.* What is recommended is that we design teaching programs that build implicit understanding of such principles into the material. This is a crucial point. Lampert's initial goal is to build implicit or intuitive (to use her word) understanding. It is not necessary that children be able to state this understanding; what is necessary is that they develop an ability to generate novel procedures for doing multidigit multiplication as opposed to something else, say, addition or even division. But to do this, it is necessary to have implicit knowledge of the requisite mathematics,

[1]Piagetians and many cognitive scientists share a constructionist perspective on knowledge acquisition. They differ, however, on the nature of mental representations that are constructed. Piaget's emphasis on the development of domain-general logical structures contrasts with the cognitive scientist's emphasis on the role of specific knowledge as a source for guiding the selection, interpretation, and storage of data.

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knowledge that serves to constrain the choice of procedures and provide a source for monitoring whatever output is generated (Gelman & Greeno, in press; Gelman & Meck, 1986; Karmiloff-Smith, 1986).

Given that an initial goal of Lampert's program is to achieve intuitive understanding, her proposal is most certainly not a variation of the New Math, which often recommended explicit teaching of the principles themselves, sometimes to the exclusion of any teaching of number facts and algorithms. The idea was that the "how-to's" would flow from explicit knowledge of mathematics. The problem, however, was that children had no idea of how to act or of what component procedures they might select and assemble into a coordinated plan of action. Even if children did have full understanding of the mathematics, they had no idea of how to put it into practice so as to satisfy the goals of getting the answer and honoring their knowledge. Lampert's proposal—that we build intuitive understanding of the principles and do so in the context of algorithm instruction—solves this problem. It does so by building-in experience in generating plans and providing materials that can be used to build intuitive mathematical understanding. And it works! Children do master techniques for getting answers, and they understand how to apply these in novel settings and with multidigit problems.

As soon as we acknowledge that we have to teach algorithms, some will ask: Why bother with so complex a program as Lampert's? After all, if we want children to know how to get the right answer, why not simply make sure they have enough trials mastering the material put in front of them? Siegler and his collaborators even provide evidence that this might work because the more trials spent on a given multiplication fact, the more likely one is to master it and hence retrieve it correctly from memory (Siegler, 1986; Siegler & Schrager, 1984). Further, Siegler's association models seem to do a rather good job of modeling the knowledge young children have (or do not have) of the arithmetic problems in addition, subtraction, and so on, found in many textbooks.

My concern with focusing a teaching program exclusively on the opportunity to have enough corrected practice with the facts of multiplication and the standard algorithm for doing multiplication is threefold. Like Lampert, I do not reject the need to practice the basic facts. Indeed, I agree with Lampert's suggestion that it may be a very good idea for children to spend some time in such activities. But as Lampert notes, it is one thing to master the single-digit multiplication facts and quite another to be able to do multidigit multiplication. Second, once one enters the arena of learning algorithms to do multidigit multiplication, one could master the procedures by rote. But doing so holds no guarantee that such learning will support the ability to solve novel problems, to understand why alternative solutions should come out with the same answers, and so on. In short, the odds are against observing transfer in the absence of a representation of the knowledge involved (Brown, Kane, &

Echols, 1985). Put differently, the data that can be explained by a learning-by-rote, association model are limited to those that index whether learners have mastered familiar materials and procedures. To be sure, there are many such indices, including number of trials to criterion, number of errors, tendency to use known strategies, solution types to known problems, rate of retrieving the correct answer, tendency to generate interfering stored facts, and so forth. Notice, however, the absence of a class of learning indices: those requiring that children succeed on novel problems, problems they have never seen before, and problems that require the generation of solutions they have never seen before. But it is the fulfilling of just such criteria that allows us to conclude that children understand what they do.

There is another reason to object to the idea that we should focus exclusively or primarily on building strong associations between a problem and its answer (or between a procedure and its appropriate setting): Such a recommendation presupposes that this is the way children learn all things. But it is one thing to say that a model of learning works because it can account for the data on hand (e.g., the error rates for different number facts and the frequency with which these are encountered at home or school) and quite another to conclude that that is the model of how children must (or will be inclined to) learn if given an opportunity to learn in a different way. I suspect Siegler's models of different early arithmetic skills do as well as they do because many, if not most, children in this country learn according to a regime that is either implicitly or explicitly based on an underlying association model. The emphasis seems to be on teaching children to get the right answer by memorizing number facts and solution paths by rote. But this is just another way of stating Lampert's objection to the standard curriculum because it follows that children will not be able to recognize novel solutions, let alone generate any themselves. Likewise, association models predict that children will err on novel problems (Briars & Siegler, 1984) and produce rather random as opposed to systematic errors (i.e., errors that are related to some implicit knowledge of mathematical principles). But these predictions are contradicted by Lampert's findings and by an ever-growing body of data. It is mainly because children are able to generate systematic, rule-related errors in a variety of learning settings that many researchers have adopted the constructionist framework of learning (see Brown, Bransford, Ferrara, & Campione, 1983, for a review of this development).

How should we characterize learners who do gain insight into the structure or principles of a domain? I have already hinted that the Piagetian framework provides one answer to this question, especially as developed by Karmiloff-Smith (1986). Like Siegler, Karmiloff-Smith characterizes the learner as strategic. Additionally, she demonstrates that children create their own solutions and supporting environments. They do so in the name of developing structured representations that mediate both understanding and problem solving in a domain. One characteristic of such learning is that children will give up a solution that works (and which must therefore have been reinforced) and will do so when they gain some, albeit incomplete, understanding of the principles that organize the class of problems they are working on. This leads them to formulate hypotheses about possible solutions and even to accept these hypotheses despite the fact that they fail to account for all the data. With enough negative examples—examples they often generate themselves (and indeed may have to)—children move on to formulate more inclusive hypotheses until they master the problem space—be it about balance, semantic or syntactic rules, or (I add) mathematics.

Although the evidence is building that children participate heavily in their own cognitive development at times, we are just beginning to map the conditions for guaranteeing this. Further, no matter how often children might do this, there is no assurance that they will find the relevant materials. Hence, the message of the young learner as active participant should not be misinterpreted to mean that we can leave children to their own devices. Our challenge is to develop the materials and teaching techniques that guarantee children will engage their participatory learning style. Lampert's pupils' ability to master multidigit multiplication stands as a significant demonstration that we can succeed at wedding such theoretical developments with practice. We need more efforts like this. It is one thing to test new ideas in a controlled laboratory setting and quite another to show how to implement them in the classroom.

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