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THE NATURE AND DEVELOPMENT OF EARLY NUMBER CONCEPTS[1]

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I. Introduction

The purpose of this paper is twofold: (1) To provide a conceptual framework within which to study the early development of concepts of quantity; and (2) to demonstrate how this framework can be used to gain insight into the young child's conception of number. Particular emphasis is placed on the problem of studying number concepts in preschool children. Although much of the paper will deal with the development of the concept of number, it is thought that the general approach can serve as a working model for an understanding of the acquisition of other quantity concepts.

A. Basic Concepts: Estimator and Operator

We begin by drawing a distinction between the process of extracting an *estimate* of a given quantity and the process by which one judges the consequences of transforming a quantity. The cognitive processes by which people determine some quantity, such as the numerosity of a set of objects, are termed *estimators*. The cognitive processes by which people determine the consequences of transforming a quantity in various ways are termed *operators*. Thus, the distinction is between processes that are used to determine how many items there are in an array or the relative numerosity of two arrays as against the processes used to determine whether transformations performed on a set affect its numerosity.

There are several reasons for distinguishing between operators and estimators. First, estimators may be thought to involve a "lower" level of processing than operators. Estimators are closely tied to perceptual input, operators are not. Operators can, at least in principle, specify the outcome of applying a particular operation to a given amount whether or not the amount is actually present. For example, adults assume that pouring a quantity of water from one container to another will not change the quantity, whether or not they are actually witnessing such a transformation of some quantity of water. In contrast, they cannot estimate accurately some quantity of water unless water is actually present to be estimated.

Second, operators may be thought to be "more cognitive" than estimators. Operators provide integrative links between successive estimates. Thus, for example, a perception that five rabbits went into the hat, in conjunction with the perception that only two rabbits came out of the hat, provides surprise through the mediation of a cognitive operator which specifies that placing arrays in hats and taking them out again are not transformations that alter number. In this case, two successive estimates of number are integrated by means of a cognitive operator.

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A third reason for regarding the distinction between estimators and operators as important is that operators are more central to a mature conception of number. Consider the category of transformations which involve displacements or rearrangements. Adults, who treat number operationally, do not believe that the numerosity of a set is altered by such transformations, no matter how great the perceptual changes produced by them. And, as Piaget (1952) has shown, young children give evidence of believing that such transformations alter the quantity. In Piaget's view it is the presence of operators in the adult that enables him to judge numbers as invariant under such transformations. Similarly, Piaget argues that young children fail to judge number as invariant under such transformations because they lack the appropriate operators. And because they lack the appropriate operators they are said to have not yet developed a concept of number. The centrality of operators in Piaget's theory can be seen from the fact that children are said not to have a concept of number until they pass the operator task, i.e., conservation. Yet most children can accurately *estimate* smaller numbers by counting well before they pass the conservation test. Piaget's assumptions are reflected in several scalogram analyses of the number concept which assign a higher developmental ranking to operational tasks than estimation tasks (Siegel, 1971; Wohlwill, 1960).

Consideration of experiments designed to investigate the development of operators highlights a fourth reason for distinguishing between operators and estimators. It appears that the way in which an individual estimates at any given time influences the way he behaves in such experiments, in particular, whether he uses operators. This observation leads to the thesis that the use of the two types of processes is related in a complex fashion, and that a consideration of how a particular estimation strategy process influences the use of operators would provide guidelines for studying the development of number concepts in young children.

To illustrate that the use of one estimation strategy as opposed to another is related to the use of operators, we start by considering a hunter who sees a flock of ducks spread out in the sky. When he first sees the flock, he estimates the number of ducks at 150. After watching the same

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flock land close together on the small pond nearby, he changes the estimate to 75. In this particular situation, the ducks simply rearranged themselves. No ducks flew away. Yet the effect of the rearrangement led the hunter to say there were fewer ducks. Why? One explanation is that he thought the number actually changed. Yet questioning the hunter would probably show he did not think this. Instead, he would say that his original estimate was a guess based on the extent of the array and that the landing of the ducks led him to alter his initial guess. Further, he would argue that this was a reasonable thing to do since he had no reason to be confident about his initial estimate. However, had the hunter reason to be confident about his initial estimate, he would not have changed it. Thus, for example, if he could have counted the ducks or been told by someone what the actual number of ducks was, he would not have suggested that there were fewer ducks after they rearranged themselves.

The example serves to illustrate that whether operators are revealed in behavioral reactions to transformations depends on the type of estimator invoked. When an estimate is based on a property that is not invariant across transformations of a set, the individual may treat each distribution of the set as an independent estimation trial. When the estimate depends upon numerosity *per se* then rearrangements may be ignored or deemed irrelevant. In this case, the use of invariance operators can be inferred. Thus, a consideration of estimation processes is central to our interpretation of the behavior of an observer who witnesses transformations of an array. Failure to consider these may lead one to conclude that the hunter lacked number invariance operators or that he treated rearrangement as a transformation that is relevant to number. These considerations point to the possibility that, to a large extent, the *apparent* absence or presence of number invariance operators may depend upon the absence or presence of the ability to estimate number with precision and confidence. If one does not estimate on the basis of number *per se*, then one cannot utilize the operator part of his number scheme. Rather, one may treat the successive presentations of different arrangements of the given quantity as separate occasions for estimating.

This argument suggests several hypotheses with respect to the development of number concepts. First, the quality of estimation strategies may determine whether a child reveals a competence for treating number operationally. Second, as processes for estimating number with precision develop, so may the use of operators. Related to both of these hypotheses is the possibility that a study of processes which contribute to a child's confidence in his estimates may aid our understanding of developing quantity concepts.

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In order to assess these hypotheses it is necessary to study the development of estimators as well as operators. In general, these two processes have not been considered together in investigations of developing number concepts. The interest in the development of number concepts can be traced back to the end of the nineteenth century. Initially, psychologists tended to investigate processes that are classified as estimators in this paper. Thus, much of the early work focused on how children count, perceive, and discriminate numbers. Although there was some discussion of operations, particularly by McLellan and Dewey (1896), Piaget's treatise on number ushered in a concerted effort to investigate operations. The earlier work did not place questions of what children knew about addition and subtraction into the framework of assessing development of operators but rather into a catalogue of number facts which children of various ages knew. Thus, for example, Brownwell (1941) assessed children's knowledge of number combinations such as $(1 + 2)$ or $(5 - 3)$ and provided normative charts showing which of these number facts children knew. The studies were not designed to determine the operations which children thought did or did not change number. Piaget shifted the framework from one in which the child's responses to number were thought to indicate mastery of number facts to one in which the child's responses to number were thought to reveal the functioning of underlying cognitive operators. Thus, further studies of children's comprehension of addition and subtraction (e.g., Smedslund, 1966a, 1966b, 1966c; Wohlwill, 1960) were placed in the context of Piaget's treatment of number as a manifestation of logical thinking.

Although most of the research that is relevant to the discussion of estimators predates Piaget's work, some work in this area has continued (e.g., Beckwith & Restle, 1966; Potter & Levy, 1968). And, as indicated below, some of the current research on the pre-conserver's notion of number is relevant to the discussion of estimators (e.g., Gelman, 1969a; Mehler & Bever, 1967; Wohlwill, 1960; Zimiles, 1963, 1966). Still, there has been little tendency to consider the two problems together and to take into account possible interactions.

The body of this paper begins with a review of the literature that pertains to estimators and a consideration of methodological problems in determining the properties upon which the child's estimate is based. There then follows a report of research on estimators from our own laboratory. Then we take up the problem of confidence in estimates. Finally, there is a selected review of previous research on operators and a summary of our current studies on operators that have been designed in light of the results of our estimation studies.

II. Estimators

A. Methodological Considerations

Estimator tasks are those tasks which were designed to determine under what conditions subjects can abstract the precise or approximate numerosity of a perceptual array. In such tasks, subjects are asked to judge, construct, or compare arrays which vary in numerosity and other attributes.

Various tasks of this kind have been used with children. For present purposes, it is necessary to decide whether the child's responses in these tasks were based on the numerosity or on other distinctive cues in the array(s). Knowing which cues determined the responses will help one determine what estimation processes the child utilized. Therefore, the question is what methodologies can or cannot be used to assess the extent to which young children respond to numerosity *per se*. An analysis of the tasks used by various investigators to study number concepts in children helps answer the question.

Some investigators use a discrimination paradigm, in which children are to respond differentially to two arrays of equal or unequal number. In some of the number discrimination studies, children are required to learn to pick the larger array (e.g., Long & Welch, 1941; Wohlwill, 1962). In others, they are asked to decide whether the arrays contain the same number or if one has more or less than the other (e.g., Rothenberg, 1969; Russell, 1936; Zimiles, 1966).

Generally, the above discrimination tasks do not indicate which attribute of the arrays was responded to by the child. Consider what interpretations can be made of "correct" responses to differences, i.e., reaching a learning criterion of choosing the array which has more (or less) objects. Figure 1 presents schematic samples of pairs of arrays that have been used in various experiments. In some cases these are placed side by side (e.g., Long & Welch, 1941; Russell, 1936); in other cases one above the other (e.g., Rothenberg, 1969; Zimiles, 1966). The actual numbers in each stimulus pair have been varied, e.g., 3 vs. 5, 6 vs. 10, 1 vs. 6, etc. Figure 1 shows that each pair of arrays always has at least one other difference redundant with the difference in numerosity. This is necessarily so when only two stimuli are used. Consider the Pairs A and B in Fig. 1 where the two arrays each have different numerosities but no differences in size, shape, or color of the objects or the pattern of dispersal. If the two arrays are of equal length (Pair B), then they must vary in density and the more dense stimulus must be the more numerous. Conversely, if density is held constant (Pair A), length varies with numerosity. In stimulus pairs C and D length and density are in conflict, respectively, with numerosity (cf. Zimiles, 1966);

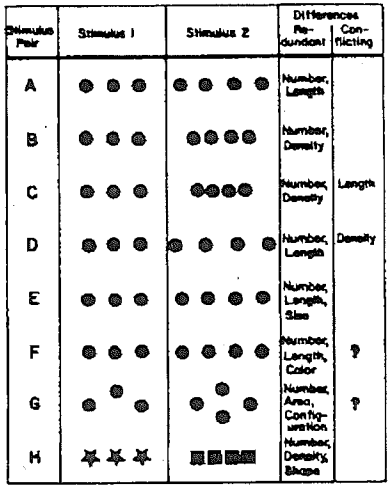

Fig. 1. Schematic illustration of representative arrays used in discrimination of number tasks where the numbers differ.

however, density and length, respectively, are redundant with numerosity. Thus, for example, if Stimulus 2 in Pair C is chosen as "more," one cannot be certain as to whether density or numerosity prevailed over length. Thus, children may score correctly without responding to number.

A related problem occurs when the number of objects in both arrays is equal. Figure 2 shows eight stimulus pairs. Each array in a pair has the same number. Inspection makes it clear that many stimulus pairs can be judged "same" on grounds other than number. Only the arrays of Pair H are alike on number alone; and this is true only if the child responds to the whole array as opposed to parts of arrays. If he does the latter, he could judge Pair H the same because they both contain a small black circle. Thus, in many cases a child could again score correctly without responding to number. Other cues are available.

The presence of redundant non-numerical cues confounds the interpretation of performance in other number tasks as well. A child required to reproduce a given set of objects for number (Brownwell, 1941; Buckingham & MacLatchy, 1930) may simply pair off each object without assessing the number (Piaget, 1952). Or a child required to identify, for example, which of three arrays contains 10 objects (Brownwell, 1941; Grant, 1938; Woody, 1931) may solve the problem by remembering that 10 means a lot and by looking for the biggest array (Brownwell, 1941). Although both

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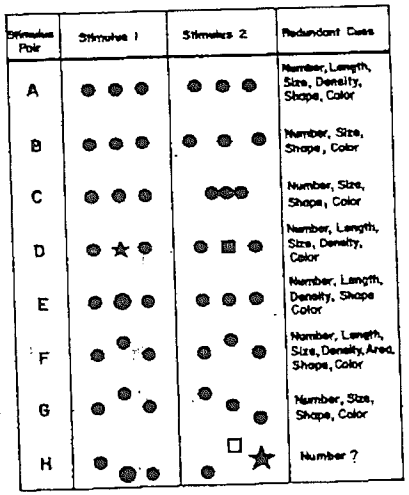

Fig. 2. Schematic illustration of representative arrays used in discrimination of number tasks where the numbers are the same.

of these strategies are "quantitative" to some degree, neither reveals the unambiguous use of number alone.

The difficulty in inferring whether a child responds to number in the preceding tasks is reduced or disappears when other response criteria are included. For example, verbal protocols may serve this end. The child might be asked why x has more, why x equals 10, or why x is the winner. In a study described later, the child's ability to respond to number was assessed with a modified discrimination task. Children aged 3–5 years were shown two arrays of toy mice differing in number (2 vs. 3) and length *or* density. The child had to determine which array was the "winner" and which the "loser." During the learning, children were asked why they identified the winner or loser as such. The majority of the children said that one had 3 or the other hand 2 or both. No child said that one was longer or more bunched up.

The use of verbal protocols is not the only way to render a response unambiguous in a number task. Baldwin and Stecher (1925) noted that children who could construct a set equal in number to that in a sample, counted extensively. Observations of this sort also serve to remove ambiguity.

Not all estimation experiments present a problem of interpretation. Many have been carried out in such a way that number was the only possible cue for consistent responding. For example, Beckmann (1924) used

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a number production task in which the children took out N marbles (or other objects) from a box. Since the experimenter in this task did not first make a similar sample, the child had no set to match. Therefore, it is unlikely that the children could have succeeded in this task without responding to the numerosity *per se*. Beckmann (1924) also used a simultaneous discrimination task with repeated trials. This involved presenting different examples of two numbers over a series of trials. The configurations and materials within and between the sets varied but the numbers did not. He scored children as having mastered the task only if they answered correctly regardless of configurations or materials. Thus, consistently correct responses must have been based on number, the only cue which was the same in all arrays.

A repeated trials design has also been employed in recognition experiments in which children have to identify the number of dots in arrays with varying patterns of dots (e.g., Brownwell, 1941; Freeman, 1912). Buckingham and MacLatchy (1930), in what they called a number identification task, instructed their testers to cast a given number of objects on the table before the child and then to ask how many there were. They reasoned that doing this three times for each of 1,356 children on each of nine numbers would serve to control for the use of any non-numerical cue.

This methodological discussion highlights the criteria used in selecting for review, in the next section, studies from the literature on the child's use of number. The review concentrates on those studies which reveal with some certainty whether or not the children were responding to numerosity. The experiments had to satisfy either of the following criteria: (*a*) the task was designed so that a correct response had to be based on the use of number, and (*b*) verbalizations or some other behavior of the child made it possible to decide which cues controlled the child's response.

B. Review of the Literature

1. *Studies with Preschool Children*

Monographs by Beckmann (1924) and Descoeudres (1921) contain some of the most extensive data on the ability of 2- to 6-year-old children to estimate number accurately. Baldwin and Stecher (1925) used Descoeudres' procedures with an American sample of 3- to 5-year-olds. Unfortunately, their data are reported in a way that makes it impossible to determine how closely these studies agree. Beckmann used four different tests of the child's ability to estimate number. The easiest was the number production task described in the preceding section. Despite its simplicity, it required a child to respond to number in order to succeed. Descoeudres also tested children with a number production task. However, in her test,

the experimenter always constructed samples against which the child could compare his own. Despite some resulting ambiguity, her results are strikingly similar to Beckmann's. It seems safe to assume that the children in Descoeudres' study were in fact responding only to number. Together, these two studies surveyed 663 children. Beckmann deliberately attempted to include children from all social strata and environments in and around the city of Göttingen. His sample included children from orphanages and day-care centers for working mothers as well as children from private kindergartens and nursery schools. Although Descoeudres does not give details of her sample, it apparently included a considerable range. The normative data from these two studies on the ability to produce a specified number accurately are shown in Table I.

The percentages in these tables represent minimum figures, not average performance. In order for a given child to be counted as capable of reproducing 2, for example, the child had to perform perfectly over several trials with a variety of materials. Table I clearly shows that children between the ages of 2 and 3 years can generally produce the number 2 with considerable reliability. They seem, however, unable to produce the number 3 reliably. At some time between the age of 3 and 4 most children become capable of producing the number 3. But it is not until they are 4½ or 5 years old that they can produce larger numbers.

The similarity in the results of these two studies serves to emphasize another theme that emerges from the Descoeudres monograph: Numerosity is a very salient feature of an array for a child, *provided the array is sufficiently small so that the young child can accurately estimate its*

TABLE I

PERCENTAGE OF CHILDREN IN THE BECKMANN AND DESCOEUDRES STUDIES ABLE TO REPRODUCE EACH NUMBER CONSISTENTLY

Sample Size			Number									
			1		2		3		4		5	
B[a]	D	Age	B[b]	D	B	D	B	D	B	D	B	D
20	5	2–0	—	40	30	40	0	0	0	—	0	—
20	19	2–6	—	79	70	74	0	16	0	0	0	0
46	21	3–0	—	100	70	100	20	19	4	4	0	4
25	30	3–6	—	100	84	97	20	67	12	13	4	0
41	36	4–0	—	100	90	97	63	78	39	25	17	11
42	31	4–6	—	100	99	100	83	87	55	61	36	32
56	27	5–0	—	100	100	100	82	96	64	81	45	33
60	15	5–6	—	100	100	100	93	100	87	73	70	14
155	14	6–0	—	100	100	—	96	100	92	100	74	93

[a] B and D stand for the authors of each study.
[b] Dash indicates data not collected.

numerosity. Descoeudres went to great lengths to avoid any mention of number, yet she obtained very much the same results as Beckmann, who required the children to produce a specified number. When the children reproduced the experimenter's arrays for Descoeudres, they did so on the basis of numerosity even though their attention was not specifically directed to this property. Thus, it appears that a young child will attend to numerosity as opposed to other cues in an array if he has the capacity to estimate the number represented.

Both Beckmann and Descoeudres demonstrate that the ability to estimate number in these and other tasks is not stimulus or task specific. Beckmann, for example, trained some children to estimate reliably the number they first failed on in the normative task. Although the children were trained with a single type of stimulus material, such as marbles, their estimating ability generalized to a wide range of other objects, including objects portrayed in pictures.

Descoeudres found that the ability to discriminate arrays consistently on the basis of number is closely related to the ability to produce numbers. Table II shows a notable drop in discrimination performance when numbers greater than 3 are represented in the arrays to be discriminated. A similar effect was found in the production tasks in Descoeudres' and Beckmann's studies (see Table I). Thus, young children can differentiate sets on the basis of number if the numerosities displayed fall within a range they can estimate.

The effect shown in Tables I and II, a sharp drop in preschooler's performance when numbers greater than 3 are used, was found throughout the Beckmann and Descoeudres studies. The generality of this effect led Descoeudres to refer to what she called the "1, 2, 3, *beaucoup*" phenomena in preschool children. She used this description to indicate that the preschool child's ability to estimate number breaks down rather abruptly at

TABLE II

PERCENTAGE OF CHILDREN WHO CONSISTENTLY DISCRIMINATED ON THE BASIS OF NUMBER IN DESCOEUDRES SAMPLE[a]

	Age in years							
Discrimination	2–6 (13)[b]	3 (17)	3–6 (15)	4 (25)	4–6 (17)	5 (13)	5–6 (8)	6 (2)
1 vs. 2	23	59	67	76	100	77	100	—[c]
2 vs. 3	23	41	67	64	100	77	100	—
3 vs. 4	0	0	13	8	5	38	75	100

[a] From Descoeudres (1921, p. 278). With permission of author's publisher.
[b] Number in brackets indicates the size of each age group.
[c] Dash indicates data not collected.

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some number between 2 and 5, most typically after 3 in children younger than 4 years of age. A typical pattern of performance for a 3½-year-old child was as follows: The child could estimate the numbers 1, 2, and 3 with reliability in a variety of tasks; he could estimate the number 4, but only imprecisely and unreliably, e.g., the child sometimes produced five or three objects instead of four. For numbers greater than four the child was grossly inaccurate and unreliable. Descoeudres introduced verbal protocols and other observations to demonstrate that the children regarded the larger numbers as equal to "a lot" and therefore undifferentiated from one another.

The generalization regarding the abrupt breakdown of performance at some number between 2 and 5 was true for all tasks in Beckmann's and Descoeudres' studies. However, the percentages of children passing a task for a given number varied considerably from task to task, particularly in Beckmann's studies. Many of the failures were due to extrinsic factors. For example, performance on Beckmann's most difficult task clearly depended on the child's ability to focus on one array of small dots shown on a paper with 11 other dot patterns. Because of these extrinsic factors, Beckmann's four estimation tasks were not scalable. The order of task difficulty for individual children could not be predicted from the group data. Beckmann's data somewhat underestimate the preschool child's ability to estimate small numbers. Since each of his tasks was an unambiguous test of the child's ability to respond to number *per se*, success on any one task indicated some such ability. Beckmann, however, does not give composite data indicating the percentage of children in each age group who passed at least one of the tests.

2. *Studies with School-Aged Children*

Several studies with school-aged children contained estimation tasks which allow one to determine whether children did or did not abstract numerosity from an array. Counting was discouraged in some of these studies but not in others. The latter type is emphasized in this section.

Buckingham and MacLatchy (1930) tested 6- to 6½-year-old children who were just entering the first grade and who had no previous instruction in arithmetic. They used a variety of number knowledge tasks, one of which required the children to estimate the number of objects thrown randomly on a table. Sets of 5, 6, 7, 8, and 10 objects were each thrown three times. Another task with the same numbers was like Beckmann's production task and required the children to hand an $N = x$ to the experimenter. Seventy percent of the children successfully estimated 10 objects on *at least one* of the three trials. The respective percentages for the numbers 8, 7, 6, and 5, were 72, 74, 75, 81. However, the ability to estimate these numbers reliably was considerably less. Thus, only 42, 45, 46, 52, and 63% of the children correctly estimated the numbers 10, 8, 7, 6, and 5 on all three trials. Similar results were reported for the production task; for example, 85% of the children produced the number 5 at least once, but only 64% produced it all three times.

Brownwell (1928) tested first graders on their ability to indicate how many dots were present in arrays varying in number from 3–12. A given number of dots was arranged in five different two-dimensional patterns and one linear pattern. The average percentage of correct responses for the numbers 3–12 is given in Table III. Freeman (1912) had children report the number of dots they saw in a briefly exposed array. The arrays also varied in number and pattern. Freeman reported that his younger subjects (6- to 7-year-olds) could grasp the number in an array as well as older children and adults when the number represented was 4 or less. As the number increased "beyond the scope of attention" performance fell off markedly for the younger children but not for the older children and adults. He attributed this difference to the older subjects' propensity to group the objects, even when no obvious pattern was present. The older subjects used more sophisticated estimation strategies. The 6- to 7-year-old subjects did not do as well as Brownwell's subjects of the same age, probably because Brownwell's subjects were not discouraged from counting whereas Freeman's were—a point to which we will return.

Brownwell (1941) used tasks comparable to those of Buckingham and MacLatchy and obtained similar results. Thus, for example, 54% of 631 6-year-old school entrants were able to put "tails on nine of the rabbits"

TABLE III

AVERAGE PERCENTAGE CORRECT RESPONSES GIVEN TO THE NUMBERS 3–12 BY BROWNELL'S (1928) SUBJECTS

Number	Average percentage correct[a]
3	98
4	95
5	93
6	91
7	85
8	81
9	78
10	78
11	70
12	69

[a] The computations summarized here were made by the present author and are based on the data given for two-dimensional patterns. Insufficient information was given for the linear arrays.

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in a picture displaying many rabbits and 80% were able to draw five marbles on an answer sheet. When instructed to give the experimenter a particular number of objects, 85, 80, 81, 80, and 77% of the children could produce 5, 6, 7, 8, and 10 objects. This finding is almost identical to that reported by Buckingham and MacLatchy for the first trial of a comparable production task.

Together these data suggest that first- and second-grade children can abstract the numerosity of arrays of 5 with no difficulty and can do so with considerable accuracy for numbers up to 9 or 10. Their ability to estimate accurately does decline as the number becomes still larger; however, this decline is not nearly so abrupt as that described for younger children.

A review of the literature on number estimation in young children has revealed two points that will reappear in subsequent sections: (1) Young children, particularly preschoolers, have difficulty estimating accurately, i.e., abstracting the numerosity of an array when the numerosity is greater than 3–5. When the numerosity is 3 or less even 2½-year-olds can frequently abstract it. (2) Provided the numerosity of an array falls within the range that a child can estimate, the numerosity is a very salient property of the array. That is, the child will attend to this property in comparing arrays even when many other readily perceived cues are also presented by the arrays. It would seem, then, that studies of the young child's ability to use number operators should involve small numbers.

C. The Role of Counting in Estimation

A considerable amount of evidence supports the hypothesis that counting[2] is the preeminent mechanism used by young children to estimate numbers of all sizes, with the possible exception of 1 and 2. It might be expected that the ability to apprehend small numbers is mediated by a "perceptual" mechanism and that this mechanism is the initial estimating process used by young children. This perceptual mechanism might rest on some apprehension of the geometric properties of the configurations made by small arrays or some at yet unexplicated process. Direct apprehension of the numerosity of small arrays without the mediation of counting is common in adults. In the literature on adult performance, it is referred to as subitizing (Beckwith & Restle, 1966; Neisser, 1966). The ability to subitize small numbers also appears in preschool children. But, as shown in Table IV, it

[2] A number of investigators have distinguished between the ability to repeat a number sequence by rote and the ability to coordinate the numerals with successive items. The term counting has the latter meaning here.

TABLE IV
Percentage of Children in Beckmann's Study who Counted or Subitized when Estimating Numbers[a]

		Age of children				
Number	Process	4–0 (48)[b]	4–6 (26)	5–0 (27)	5–6 (20)	6–0 (24)
2	Count	25	17	11	4	2
	Subitize	75	83	89	96	98
3	Count	71	33	35	21	5
	Subitize	29	67	66	79	95
4	Count	80	80	71	52	17
	Subitize	20	20	29	48	83
5	Count	100	88	76	61	33
	Subitize	—	12	24	39	67
6	Count	100	96	90	64	41
	Subitize	—	4	10	36	59

[a] From Beckmann (1924, Table 8, p. 28).
[b] Number in brackets indicates the size of each age group.

appears to develop *after* children have learned to estimate a number by counting.

Table IV shows the results of Beckmann's analysis of the way in which children arrived at an accurate estimate of the number of items in an array. Depending on their behavior during the task and their explanations of how they reached an answer, Beckmann categorized the children as counters or as subitizers.[3] Children classified as counters were observed to count before giving their answer and said they counted when asked how they knew an answer. Children classified as subitizers responded very quickly without giving any indication of counting and said they could see when asked how they knew the answer. Thus, for example, children said "It looks like two," "I can see it's two." It can be seen in Table IV that the younger the child, the greater the tendency to count for all numbers. Furthermore, the larger the number, the greater the tendency for all children to count. Together these results support the conclusion that children estimate a number by counting before they can subitize the same number.

The same pattern of development from counting to subitizing appears in Beckmann's production task. In this task children had to produce a designated number of marbles from a box. Beckmann reported that the children used three strategies in this task: (1) Children counted as they removed one marble at a time; (2) children looked at the marbles and then pulled out

[3] The term subitize is used as a translation of Beckmann's term Erkennen.

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TABLE V
PERCENTAGE OF CHILDREN IN BECKMANN'S PRODUCTION TASK WHO USED THE DIFFERENT STRATEGIES[a]

Strategy	Age level					
	3–0/3–6	4–0	4–6	5–0	5–6	6–0
Count	58	57	46	39	43	48
Look and take	33	25	30	30	30	32
Group	8	18	24	31	27	20

[a] From Beckmann (1921, Table 6, p. 25).

the appropriate number; (3) children constructed the sum taking two groups of marbles, e.g., a child asked to produce five marbles first took two marbles and then another three. The percentage of children who used each of the strategies appears in Table V. Counting was the prevalent strategy used by children of all ages. The older children used the grouping strategy more than the younger children.

Brownwell (1928) and McLaughlin (1935) reported a similar effect. As indicated above, Brownwell tested children in grades 1 through 5 for their ability to identify the number of elements in arrays of 3–10. The arrays were presented for five seconds each, allowing sufficient time to count some of the displays. Brownwell reported that the younger children almost always counted and seldom took advantage of the patterns in the display. McLaughlin indicated that 3- to 6-year-olds typically counted in order to determine the number of objects in an array, even when the numbers were small. As the number of items a child could count increased, so did the ability to estimate. Good counters made high scores.

A comparison of performance in experiments in which children might have counted with those in which they might not have, adds support to the hypothesis that young children initially estimate by counting. We have already described Brownwell's (1928) and Buckingham and MacLatchy's (1930) work on estimation. Brownwell used a dot recognition task. The comparable Buckingham and MacLatchy test involved showing children the result of a random throw of objects. In both of these studies, children were *not* prevented from counting. As indicated, Brownwell reported that the 6-year-olds did count. Douglass (1925) used three number tasks with children of different ages, including 6-year-olds. Two of his tasks were similar to Brownwell's and Buckingham and MacLatchy's. In one, children saw 1–10 blue dots arranged in a row on cards; in the other, children had to identify the number of pennies (1–10) held in an open hand. The third task involved choosing the one of ten cards displaying the number of dots which corresponded to a spoken word between one and ten. In all three tasks, stimulus exposure was brief and the children were discouraged from

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counting. Although the three studies had samples of varying ages, all included a 6-year-old group. A comparison of Douglass' 6-year-olds with equally old children in the other two studies reveals a large discrepancy. In the studies in which the children were allowed to count, approximately 70% accurately estimated nonlinear arrays of ten on at least one trial. And although children in Brownwell's study were less likely to estimate linear arrays of 9, 10, and 11 accurately, 54% did so.[4] In general, these studies give the impression that 6-year-old children can estimate arrays of ten items with reasonable accuracy. In contrast, in Douglass' study in which children were not allowed to count, only 8% of the children could estimate the numerosity of ten element arrays. Such a low score means the children did poorly on all three of Douglass' tasks, including the two which appear comparable to those used in the other studies. Although the studies differed in a variety of ways, some of the discrepancy in estimation scores may reasonably be attributed to the presence or absence of counting.

Errors in estimation may reflect errors in counting. Russell (1936) reports that some children erred because of counting mistakes when asked to judge which of two groups of blocks had the "most." Thus, for example, a child counted an 8 vs. 9 display of blocks as 8 vs. 8 and accordingly reported the displays had the same number.

Counting may not only be the first mechanism used in estimation but may also be the basic mechanism used when children begin to add and subtract. Various investigators report that the initial stages of adding and subtracting involve counting (e.g., Beckmann, 1924; Brownwell, 1941; Ilg & Ames, 1951; Reiss, 1943). Thus, for example, a child may add 8 and 3 by counting "8, 9, 10, 11" or may even count to 8 first and then proceed to 11.

To summarize, a child's ability to abstract number seems related to his ability to count. Young children tend to count even when estimating small numbers. The ability to subitize and group elements within arrays develops later.

D. AN EXPERIMENTAL INVESTIGATION OF ESTIMATORS

1. *Introduction*

The research considered above indicates whether or not a child can abstract number and provides some evidence as to how he estimates on the basis of number. It does not, however, allow one to determine the nature of estimators used by the child who does not abstract number *per se*. This question is of some theoretical interest.

[4] The manner in which data were presented for the linear arrays allows computations of a composite score only for the numbers 9–11.

One source of this interest is Piaget's writings (Piaget, 1952, 1968). These contain the hypothesis that there are four stages in the development of estimators, and that at each of these stages children use different information in the array. Piaget provides a description of the first two stages in his interpretation of Mehler and Bever's (1967) study of the ability of 2- to 6-year-old children to compare the number of objects in two rows before and after the rows are transformed. His account of the stages in the conservation of number provides a description of the last two stages. Piaget suggests that Mehler and Bever's work shows that very young children estimate on the basis of density, judging which row is more "heaped" or "crowded," while somewhat older children estimate with length cues, judging which row is longer. According to Piaget, the early use of density cues derives from a topological intuition; and the subsequent use of length cues derives from an ordinal comparison of how far each of two rows extend. This explanation is derived from research on space concepts (Laurendau & Pinard, 1970; Piaget & Inhelder, 1956), which shows that topological notions of space develop before Euclidean ones.[5]

Piaget formulates three stages in the development of number conservation. In the first stage, children center on a perceptual cue that changes with transformations, and fail to conserve; in the second stage, they begin to consider both the length and density of arrays, and sometimes conserve. However, if a perceptual change introduced by a transformation is too great, they will focus on it and fail to conserve. Finally, in the third stage, the child recognizes the compensatory relationship between the density and length of an array and can conserve number. This account suggests that in the first stage of conservation children use length or density; then they begin to use both; and finally they can abstract number. Combining the consideration of Mehler and Bever's data on very young children and Piaget's own account of the stages in conservation in 5- to 7-year-old children, we arrive at the hypothesis that there are four stages in the development of estimators. In the first, children use density; then they use length; then length and density; and finally, number alone.

A related hypothesis has been advanced by investigators who suggest that failure on a conservation task reflects lack of a set to respond to number (e.g., Bearison, 1969; Gelman, 1969a; Zimiles, 1963, 1966). According to Zimiles and Gelman young children think of number in a multidimensional fashion. The hypothesis is that there is a hierarchy of cues controlling the young child's attention to the numerosity of an array. Included

[5] However, Piaget's definition of topology appears to differ from the mathematical system of topology in which the concept of density does not appear.

in such a hierarchy are cues like length, density, arrangement features, and the numerosity of an array. With development or experience, number is assumed to become differentiated from the other cues, and thereby more salient in the hierarchy. This position implies that experiences which serve to set young children to attend to the number and not to other features of an array will increase the likelihood of their abstracting number. Another implication is that as children develop better skills for attending to number, the size of the hierarchy controlling number estimators decreases and number *per se* becomes dominant. Older children should use number more in estimating than younger children.

To evaluate these hypotheses, one needs a method that yields the likelihood of children responding on the basis of number as well as other features of the array. One possibility is a choice procedure. However, as indicated in Section II A, two-choice procedures are problematic in that they often yield ambiguous data. To circumvent the problem of interpretation presented by a two-choice paradigm, a three-choice procedure was used. In previously assessing whether children were most likely to attend to the length, orientation, or number in a row of elements, the writer (Gelman, 1969b) employed the modified method of triads used by Suchman and Trabasso (1966a, 1966b) to study color-form preferences. Children were shown a series of triadic stimulus arrays. Each triad was constructed so that every pair of arrays in a triad was alike on a value of one binary dimension (e.g., number) and different on values of two other dimensions (length and orientation). Children had to judge which two arrays in the triad were the same. Since they received no feedback, there were no right or wrong answers. It was assumed that a child's choice of a particular pair of stimuli reflected the cue to which he attended on that trial. By considering the overall frequency of number, length, and orientation responses, it was possible to determine the likelihood that each of the cues—length, number and orientation—affected a judgment.

The study reported here also involved the triad technique. The concern was to study the development of estimators and the use of length, density, or number in an estimation task. Thus, the instruction to the children was to select the two of three stimulus arrays that had the "same number." An example of triads shown to the children is given in Fig. 3. It can be seen that each pair of arrays in a triad was alike on only one value of either a number, length, or density variable. Thus, a choice of a particular pair of arrays in response to the question of which two arrays had the same number had to be based on that variable. For example, if a child chose stimulus pair A and B it would be inferred that he responded to number. If he chose stimulus pair A and C, it would be inferred that he responded to length.

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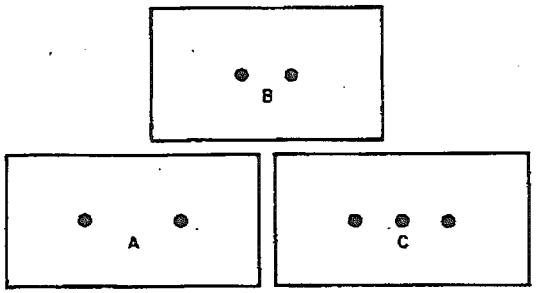

Fig. 3. Schematic representation of triadic stimulus array which varies values of only length, number, and density.

2. *Subjects*

Children who participated in this experiment were drawn from nursery and elementary schools in the Philadelphia area. Although some black middle- and lower-class children attend the nursery schools, all schools serve a predominantly white middle-class population. The design originally included 3-, 4-, 5- and 7-year-old groups. After testing fifteen 3-year-olds, who responded primarily on the basis of position, we decided to drop this age group. In all, 37 nursery schoolers (median age = 4–6 years); 45 kindergarteners (median age = 5–8 years); and 30 second graders (median age = 7–8 years) were tested. Seven of the nursery school subjects (*Ss*) were dropped because of their tendency to respond to place cues. An equal number of the remaining children from each age group was assigned to one of three independent stimulus conditions. An effort was made to counterbalance sex.

3. *Triad Construction*

There were three sets of 72 triads, one for each of three independent stimulus conditions. A triad consisted of three 12.70 × 20.32-cm cards, each containing a row of red (or green) .64-cm circles (or stars). The items in the row on each card were spaced equidistantly. Cards within each triad were placed to form a triangle.

The choice of values for the critical dimensions of length, number, and density was governed by three considerations. First, we wanted to vary systematically the values of these attributes. Second, we wanted to include comparisons of numbers as small as 2 vs. 3 in order to set our results against those of others who have studied estimation in young children. Third, we wanted to generate stimulus triads that contained one pair of arrays with an equal number, another pair with equal lengths, and a third pair with equal densities.

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The former considerations present a design problem since the critical dimensions are not independent. For example, a variation in the number of an array effects a simultaneous variation in length, density, or both. To facilitate the description of the way in which this problem was handled, Table VI presents a summary of the length, number, and density relations selected to generate the triads within each stimulus condition.

As stated previously, three independent groups of subjects were assigned to each of three conditions. The values used to generate the triads within each condition are summarized under the columns labeled with Roman numerals. The three conditions differed in terms of which of the three dimensions (number, length, density) took on a fixed pair of values for all triads used in that condition and which dimensions took on the permissible combinations of four values. Thus, in Condition I, the children saw a series of triads that all displayed the same pair of number values but varied in the pair values of length and density. Children in Condition II saw triads that all displayed the same pair of density values but varied with respect to the pair values of number and length. Children in Condition III saw triads that all displayed the same pair of length values but differed with respect to the pair values of number and density.

Within each condition there are only three permissible combinations of the number, length, and density values selected. The three permissible combinations are shown in Rows A, B, and C of Table VI. Any other combination violates the requirement that one pair of arrays in each triad should have an equal number, another an equal length, and another an equal density. Note that basic triad B was used in all three conditions. To construct the basic triads represented by the combination of value pairs shown in the rows, the smaller number of each number pair was used in two arrays, the larger number in one array. The length and density values which were used twice were dictated by this constraint. Then each of the basic triads was prepared with two colors (red or green), two shapes (circles or stars), and the six possible positional permutations of the three constituent stimuli. This yielded 24 versions of a triad. With three basic triads per column of Table VI, 72 triad versions resulted for a given stimulus condition. In order to make the results of this triad generating procedure more concrete, Table VII presents the actual lengths and densities of the triads in the first column of Table VI.

4. *Procedure*

The experiment involved three phases spread out over two days. The first phase was a pretest, the second the triad test, and the third a counting test. A child's participation each day took 15 to 20 minutes. The 12-trial pretest was an oddity task described elsewhere (Gelman, 1969a). The stimuli

TABLE VI
DESCRIPTION OF BASIC TRIADS IN ESTIMATION EXPERIMENT

Basic triad	Combination of value pairs in each condition[a]		
	Condition I	Condition II	Condition III
A	3, 5 L, $2L$ $\frac{1}{2}D$, $\frac{1}{2}D$	2, 3 $\frac{1}{2}L$, $\frac{1}{2}L$ $\frac{1}{2}D$, D	2, 3 $\frac{1}{2}L$, L $\frac{1}{2}D$, $\frac{1}{2}D$
B	3, 5 $\frac{1}{2}L$, L $\frac{1}{2}D$, D	3, 5 $\frac{1}{2}L$, L $\frac{1}{2}D$, D	3, 5 $\frac{1}{2}L$, L $\frac{1}{2}D$, D
C	3, 5 $\frac{1}{2}L$, $\frac{1}{2}L$ D, $2D$	5, 9 L, $2L$ $\frac{1}{2}D$, D	5, 9 $\frac{1}{2}L$, L D, $2D$

[a] $L = 9.0$ cm as measured from the center of end items of each row; $D = 1/2.25$ cm where 2.25 cm represents the distance from the center of one element to the next.

were small plastic toys glued on 6.35- × 7.62-cm masonite blocks. The children were asked which of the stimuli were the same, on half the trials, and which was different on the other half of the trials. Verbal feedback was given. As found in a previous study (Gelman, 1969a), this was an easy task for the children.

The triad test always began on the same day as the pretest. In this session a child was tested on 36 of the triads. The remaining 36 triads were administered in the second session. The children were given the counting test at the end of triad testing on the second day. The order in which the triads appeared was randomized for each child. Each child experienced only one of the three stimulus conditions.

TABLE VII
STIMULUS VALUES IN CONDITION I BASIC TRIADS IN ESTIMATION EXPERIMENT[a]

		Stimulus		
		1	2	3
A	Number	3	3	5
	Length	18	9	18
	Density	.11	.22	.22
B	Number	3	3	5
	Length	9	4.5	9
	Density	.22	.44	.44
C	Number	3	3	5
	Length	4.5	2.25	4.5
	Density	.44	.89	.89

[a] Lengths are measured in centimeters between the centers of the end items in each row. Density values are the reciprocals of distances in cm from the center of one element to the next; smaller values reflect lesser densities.

The triad test began by telling the child, "This game is like the toy game we played. Now you will be looking at cards which have stars or dots on them instead of toys. I want you to point to or give me the cards that have the same number." Specific responses were *not* reinforced; however, at random times, the experimenter told the child he was doing very well and offered general warm support.

The counting test consisted of asking the child to count to 10 and then select six of ten objects. The experimenter interrupted the counting if and when the child had counted as far as 10. All but one of the children could count to 10.

5. *Results*

a. *Overall proportion of number, length, and density choices.* The data were initially analyzed to yield the overall proportion of length, number, and density choices made by all *S*s within a condition. Table VIII presents these proportions for each age group and each triad subset within a condition.

Children in Condition I responded primarily to number. Only the nursery school group showed any tendency to respond to length, and there were virtually no density choices in all three age groups. Condition I reveals no effect of varying the values of length and density. Given fixed number values (3 and 5), the children were equally likely to estimate on the basis of number for all three subsets. In Condition II, the pair values for density were the same throughout and the values for length and number varied. In Condition III, the pair values for length were the same throughout and the pair values for number and density varied. Variations in the pair values of the number dimension affected both nursery and kindergarten *S*s but not second grade *S*s. No matter what values number had in Conditions II and III, second grade children always estimated on the basis of number. In contrast, the tendency for younger children to estimate on the basis of number varied depending on the numbers shown in the arrays. When the number values were 2 and 3 (Triad A in Conditions II and III), they always responded to number. When number values were 3 and 5 (Triad B in Conditions II and III), they still responded predominantly to number but not as consistently. When the number values were 5 and 9 (Triad C in Conditions II and III), even fewer choices were coded as number responses. Increasing the magnitude of the number values was more disturbing for the nursery school *S*s than for the kindergarten *S*s.

When children did not estimate on the basis of number, they primarily responded to length as opposed to density. Thus, for example, with arrays of 5 vs. 9 (Conditions II C and III C), nursery school children responded to length on 62% of the trials, and to density on only 9.5% of the trials.

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TABLE VIII
PROPORTION OF LENGTH, NUMBER, AND DENSITY CHOICES IN EACH CONDITION FOR EACH AGE GROUP IN ESTIMATION EXPERIMENT

Triad subset and summary of value pairs in each condition	Chosen dimension	Age Group: Nursery ($N = 10$)	Kindergarten ($N = 15$)	Grade 2 ($N = 10$)
Condition I				
A (3,5:*L*,2*L*: ½*D*,½*D*)	Number	.88	.99	1.00
	Length	.12	.00	.00
	Density	.00	.01	.00
B (3,5:½*L*,*L*: ½*D*,*D*)	Number	.84	.98	1.00
	Length	.16	.02	.00
	Density	.00	.00	.00
C (3,5:½*L*,½*L*: *D*,2*D*)	Number	.87	.97	1.00
	Length	.12	.03	.00
	Density	.01	.00	.00
Condition II				
A (2,3:½*L*,½*L*: ½*D*,*D*)	Number	.96	.96	1.00
	Length	.04	.02	.00
	Density	.01	.02	.00
B (3,5:½*L*,*L*: ½*D*,*D*)	Number	.75	.90	1.00
	Length	.24	.08	.00
	Density	.01	.02	.00
C (5,9:*L*,2*L*: ½*D*,*D*)	Number	.32	.68	1.00
	Length	.57	.25	.00
	Density	.11	.07	.00
Condition III				
A (2,3:½*L*,*L*: ½*D*:½*D*)	Number	.98	.99	1.00
	Length	.01	.01	.00
	Density	.01	.00	.00
B (3,5:½*L*,*L*: ½*D*,*D*)	Number	.72	.91	1.00
	Length	.24	.07	.00
	Density	.04	.02	.00
C (5,9:½*L*,*L*: *D*,2*D*)	Number	.25	.34	1.00
	Length	.67	.62	.00
	Density	.08	.04	.00

The data reveal an effect of varying the values of length and density. The nature of this effect was most pronounced in the kindergarten groups in Conditions II C and III C in which the pair values of number were the same but the pair values of length and density differed. These conditions are circled in Table VIII. The *Ss* in Condition II C responded to number more than length. In contrast, the same age *Ss* in Condition III C responded more to length than number. The description of these conditions shows that the average density of the element in the III C triads was greater than that in the II C triads, and that the rows in III C were shorter than those

in II C. Thus, an increase in densities accompanied by a decrease in lengths led the children to base their response more on length than density. The same conditions for the youngest group yielded a similar effect. A chi-square comparison of the frequencies from which the proportions were derived yielded a significant effect of condition ($\chi^2(2) = 6.4$, $p < .05$).

A comparison of all three stimulus conditions reveals in the younger groups an effect of keeping constant the values of a dimension. In Condition I, with constant values of number in all three subsets, younger children were more likely to estimate on the basis of number than were children in Conditions II and III, who saw different values of number in each subset. On the triad subset common to all three conditions, i.e., Subset B, in Condition I nursery school children responded to number on 84% of the trials; in Conditions II and III, they chose number on 75 and 71% of the trials. Similarly, kindergarten children estimated on the basis of number on 98% of the Condition I B trials as opposed to 90% of the Condition II B and III B trials. Thus, there seems to be a tendency for younger children to estimate less consistently on the basis of number when they are shown variations within this dimension. The import of this finding is discussed in Section III on estimator confidence.

b. *Analysis of individual tendencies.* Individual differences were examined to determine whether individual children estimated primarily on the basis of one dimension, and, if not, what combination of dimensions controlled their judgments. For a given subset of 24 trials, a child whose choices were based on one dimension for at least 16 of the 24 trials was judged dominant on that dimension. If no one dimension was chosen 16 or more times but two of the dimensions between them accounted for 22 of the 24 choices, a child was scored as a *length–number*, *length–density* or *number–density* responder. Otherwise, he was scored as a *mixed* responder. The percentage of children who met these various criteria is shown in Table IX. These percentages are given for each condition and age level.

Number–density and length–density percentages are not entered in the table because no child showed the length–density response pattern and only one child showed the number–density response pattern. This child was in kindergarten Condition III and he gave this response pattern only in Triad C.

A comparison of Tables VIII and IX shows that the group and individual data analyses yield the same pattern of results. Furthermore, no child estimated only on the basis of density in any condition. The density responses given in Table VIII are attributable to the mixed choice patterns in Table IX. Since the mixed choice pattern can be taken to mean no consistency in the way these children estimated, we may conclude that no children in this experiment estimated on the basis of density.

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TABLE IX
PERCENTAGE LENGTH, NUMBER, DENSITY, NUMBER-LENGTH, OR MIXED RESPONSE PATTERNS IN EACH CONDITION FOR EACH AGE GROUP IN ESTIMATION EXPERIMENT

Triad subset and summary of value pairs in each condition	Response pattern	Age Group		
		Nursery ($N = 10$)	Kindergarten ($N = 15$)	Grade 2 ($N = 10$)
Condition I				
A (3,5:L,2L: / ½D,½D)	Number	90	100	100
	Length	10	0	0
	Density	0	0	0
	Number-Length	0	0	0
	Mixed	0	0	0
B (3,5:½L,L: / ½D,D)	Number	80	100	100
	Length	10	0	0
	Density	0	0	0
	Number-Length	10	0	0
	Mixed	0	0	0
C (3,5:½L,½L: / D,2D)	Number	90	100	100
	Length	10	0	0
	Density	0	0	0
	Number-Length	0	0	0
	Mixed	0	0	0
Condition II				
A (2,3:½L,½L: / ½D,D)	Number	100	93	100
	Length	0	0	0
	Density	0	0	0
	Number-Length	0	0	0
	Mixed	0	7	0
B (3,5:½L,L: / ½D,D)	Number	70	93	100
	Length	20	0	0
	Density	0	0	0
	Number-Length	0	0	0
	Mixed	10	7	0
C (5,9:L,2D: / ½D,D)	Number	20	60	100
	Length	50	14	0
	Density	0	0	0
	Number-Length	10	0	0
	Mixed	20	26	0
Condition III				
A (2,3:½L,L: / ½D,½D)	Number	100	100	100
	Length	0	0	0
	Density	0	0	0
	Number-Length	0	0	0
	Mixed	0	0	0
B (3,5:½L,L: / ½D,D)	Number	50	100	100
	Length	0	0	0
	Density	0	0	0
	Number-Length	20	0	0
	Mixed	30	0	0
C (5,9:½L,L: / D,2D)	Number	0	26.6[a]	100
	Length	50	53.3	0
	Density	0	0	0
	Number-Length	30	0	0
	Mixed	20	13.3	0

[a] One child (6.7% of the group) gave the number-density pattern; this pattern is not included in the table, and therefore the kindergarten III C percentages do not sum to 100.

6. *Discussion*

This study was conducted to determine the extent to which young children estimate on the basis of the actual number of elements in an array. It also explored the nature of the cues to which children respond when they do not estimate on the basis of numerosity.

As in the Descoeudres and Beckmann studies, it was shown, that 4-year-olds consistently respond to number when the arrays contain two or three items. When the arrays contain three or five items, they still respond primarily but less consistently to number. Similarly, 5-year-olds always choose on the basis of number for 2 vs. 3 triads, but sometimes do not when confronted with 3 vs. 5 triads. With arrays of still larger magnitudes (5 vs. 9), 4-year-olds no longer estimate on the basis of number; and whether 5-year-olds do depends on the length and density of the displays. Second-grade children always abstract number in all conditions. Thus, 4- and 5-year-olds can abstract the numerosity of small arrays and have difficulty doing so with larger arrays.

Several features of the results indicate that children who do not respond to numerosity use length instead. First, as shown in Table IX children do not use density exclusively nor in combination with another dimension. Second, in conditions in which children sometimes respond to number and at other times do not, they shift between using number alone, length alone, and number and length alternatively. Thus, for example, in the 5 vs. 9 condition, a decrease in length and an increase in density shifted responding by kindergarteners away from number to length.

The failure to find children estimating on the basis of density or density and length in any age group or condition is surprising in the context of Piaget's theory. Perhaps still younger children would respond to density. The 3-year-old sample was originally included to provide a test of this hypothesis; however, the children of this age responded primarily on the basis of place, in spite of concerted experimenter efforts to break this set. Apparently, the task was too difficult for children this young, and some other procedure will have to be used to test the hypothesis that density is the dimension that children initially use in estimation. Still, the finding that children responded to length and number as opposed to length and density suggests that density is not a primitive dimension in the child's concept of number whereas length and number are. Perhaps when children begin to use density, they treat it as a derived attribute in their conceptual scheme of number. If so, older children should more readily respond to density than younger children. Some evidence for this hypothesis comes from a recent dissertation by Henry (1971). In a study of kindergarten and third-grade children's number concepts, Henry used a triad choice task much like that described here. Children had to choose the two of three rows of

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chips (where each chip in each row was placed one above the other) that were the same. The magnitude of the arrays was greater (7 vs. 14) and children were not specifically asked about number. Still, Henry found, in agreement with the present results, that older children were more likely to choose on the basis of number and younger children on the basis of length. Further, both groups made relatively few density responses. Probe questions designed to assess whether children had the other dimensions "available" revealed that it was primarily the older children who referred to the relative densities of the rows (e.g., "these are closer together").

Younger children in the 5 vs. 9 conditions responded more to length than number when the arrays were shorter and denser. This suggests that when number is large and the array is spread out, young children can keep track of the number. As a result, they estimate on the basis of number, either by counting or grouping the elements. If the elements are too close to each other, however, young children can no longer use number-based estimators and shift to estimating number on the basis of length. Unfortunately, the data collection method does not allow a determination of whether the children tended to count the less dense arrays more often, but subjective impression is that they did so.

Attention studies which vary the relative salience of dimensions (e.g., Trabasso & Bower, 1968) suggest that an increase in density and decrease in length would highlight density for young children and therefore increase the likelihood of their responding to it. The actual responding more to length suggests that the children attend to variations in dimensional salience only if their concept of number utilizes that dimension. If so, accounts of the development of number concepts in terms of attention alone (e.g., Gelman, 1969a) are inadequate and must include the possible nature of the child's conception of number. It appears that the cues to which a child attends depend on his prevailing concept of number.

In sum, this experiment shows that the older a child is the more likely he is to estimate linear arrays on the basis of number. As the number values of an array increases, so does the tendency for younger children to estimate on the basis of length. Finally, 4- to 7-year-old children do not estimate on the basis of density. These results suggest that the young child's concept of number is initially based on length and number and that the understanding of density as an attribute that is relevant to number develops later. This hypothesis derives neither from a consideration of the writer's own work nor from Piaget's theory. Indeed, it differs markedly from Piaget's formulation of the development of number concepts, which treats the notion of number as a derivative of the child's initial use of length and density.

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III. Estimator Confidence

One finding in the estimation study deserves attention before operators are discussed. Young children tended to use number less consistently when presented a series of triads that varied in the values of number than when presented a series of triads that had all values of number fixed at 3 and 5. Thus, in Condition I where the values of number were consistently 3 and 5, 4-year-olds responded to number on 86% of the trials. In Conditions II and III where the value of number varied, children of the same age estimated on the basis of number less often (73.5% of the trials). A similar but smaller effect appeared in the 5-year-olds, whereas no such effect occurred for the oldest children. The question is, what might account for such a finding?

In the earlier example of the hunter, it was suggested that he changed his estimate after the ducks rearranged themselves, in part because he was not confident about his first estimate. Given this lack of confidence in his initial estimate, he altered his answer as his attention was drawn to new information. The question now raised is whether lowered confidence also accounts for the tendency of younger children to respond less consistently to number when this attribute varies across trials.

To show that a young child's performance on number tasks may sometimes reflect a lack of confidence more than a lack of cognitive ability two effects need to be demonstrated. The first is that the child behaves as if he lacks confidence. The second is that altering the child's confidence by varying conditions that affect it changes the likelihood of his meeting the criterion of task success.

Assume that an individual lacks confidence in his ability to deal with a task. How might this show itself in behavior? The person might vacillate in his choice of a solution, he might change his answer with repeated questioning, or he might be inconsistent in his responses to a sequence of related questions. Thus, evidence that a young child is inconsistent in responses to quantity tasks or that he changes his answers with repeated questioning may indicate his lack of confidence in his own performance.

A study by Rothenberg (1969) reveals that young children are inconsistent in the way they answer a pair of questions about the effect of a transformation in a number conservation task. Rothenberg classified the pattern of responses of children who failed to conserve as consistent or inconsistent. Pre-kindergarten and lower-class children were classified as inconsistent more often than kindergarten and middle-class children. Rothenberg's data seem to show that children scored as inconsistent were so classified because of errors on the second question. Thus, it might be

argued that the younger children were more likely to change their answers with repeated questioning because they lacked confidence in their initial response. This possibility points to the need for research designed to assess the consistency of responses to a series of repeated questions.

As indicated above, a child who lacks confidence should be influenced by manipulations which enhance or decrease confidence. If a child is not sure he has answered a question correctly, then he should be influenced by external sources which suggest that there is another answer or that he has misinterpreted the question. An hypothesis related to this argument holds that children interpret the transformations and subsequent requestioning in the second phase of conservation tasks as a challenge to their original answer and this misperception leads the children to think that the experimenter expects them to say that the quantities are different (Gelman, 1969a; Zimiles, 1963, 1966). In this regard, Russell (1936) found children tend to focus on differences if asked a question that could be interpreted as implying a difference.

Russell reported two experiments on number discrimination abilities in kindergarten through second-grade children. Children were shown two sets of blocks, varying in color both within and between the sets. The number combinations used in each experiment are shown in Table X. The main variable differentiating between the two experiments was the form in which the number discrimination question was asked. In the first (Experiment A), children were asked several questions about two sets of blocks, including "Which pile has the most blocks?" If they appeared to have difficulty with this question, they were then asked, "Are the piles of blocks the same?" "Does one pile have more blocks than the other?" "How many blocks does this pile have?" Scoring was based on the child's ability to indicate which pile had more when there was a number difference and to say that neither had more when each set had the same number. In the second experiment (B), children were asked, "Are the piles of blocks the same? Are the piles of blocks equal in number?" A correct response involved saying that they were equal if in fact they were, or saying that they were not equal when each pile had a different number of blocks.

Table X gives percentage errors for the different number combinations in Experiments A and B. As might be expected, the children were more accurate the smaller the numbers, no matter how the question was asked. However, the form of the question did alter performance accuracy. For the combinations of different numbers, the children were more often correct when asked which combination had more; and for the combinations of identical numbers, they were more often correct when asked whether the combinations were equal. Russell concluded that when the numbers are identical, asking the child which has more focuses his attention on irrelevant

TABLE X
PERCENTAGE ERRORS FOR THE NUMBER COMBINATIONS USED BY RUSSELL[a]

Number combinations	Question type: Which has more? (Experiment A)[b]	Question type: Are they equal? (Experiment B)
Identical		
10–10	86	48
9–9	72	52
8–8	86	48
7–7	66	48
6–6	66	48
5–5	52	24
4–4	28	16
3–3	14	4
2–2	14	0
1–1	10	0
Different		
9–10	31	52
8–10	10	28
8–9	24	48
7–10	3	12
7–9	14	12
7–8	28	28
6–8	3	4
6–7	14	40
5–7	—	18
5–6	10	32
4–5	7	13
3–5	3	—
3–4	3	3
2–4	3	—
1–5	3	—

[a] Based on Russell (1936, Table 7 and 8, p. 206). With permission of author and publisher.
[b] Dash indicates data not collected.

differences between the sets, e.g., color. The few errors made even for the 3–3, 2–2, and 1–1 combinations add support to the notion that some children took the question to mean they should focus on the differences.

To the extent that a child's performance reflects a lack of confidence, feedback should either improve or depress performance, depending on the nature of the feedback. Training studies that have investigated the effects of feedback lend support to the hypothesis that failure on a conservation task may be attributed in part to a lack of confidence. Children who failed to conserve on a pretest were more likely to conserve on a posttest given after a training phase which involved feedback to focus their attention on relevant features of the conservation problem (Bruner, Olver, Greenfield,

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et al., 1966; Gelman, 1969a). Similarly, training procedures which provide explanations of the conservation principle also enhance the posttest probability of conserving (Beilin, 1965; Smith, 1968).

The estimation study described in Section II, D may also be interpreted as showing that external sources which suggest another answer will influence children who lack confidence. Younger children in this experiment were less likely to respond to number in a 3 vs. 5 triad if they also were tested on 5 vs. 9 triads. Since younger children tend to judge the latter triads on the basis of length, those trials may serve to suggest length as the "correct" cue. A child who is not sure that the task requires responses based on number might shift his basis of responding in the face of such an experience. Older children are apparently more confident; the effect of varying the magnitude of the arrays decreased with age, and second-grade children were not influenced at all by this manipulation.

Where confidence is an issue, the opportunity to succeed at the beginning of a series of questions should affect subsequent performance. Without external feedback, the initial performance might set the pattern of responding. Zimiles (1966) found that the way in which children began to answer a series of related conservation tasks influenced the mode of responding to subsequent tasks in the series. If children were given first a number conservation task with three items in each array and then one with a larger number of items, they were more likely to conserve with both small and large arrays than if they were initially tested with large arrays. In the context of the present discussion, it can be argued that the small-large sequence maximized the likelihood of the children being confident that they were to respond to number *per se*. The previous section showed that 5-year-old children firmly base their estimates of arrays with three elements on number *per se*; however, arrays of nine tax their ability to do so consistently. Accordingly, children of this age should be more confident when confronted with small arrays than larger ones. Thus, starting a child on small arrays should increase his chances of confidently answering the initial question correctly. If so, this should increase his chances of continuing to do so for larger numbers, which, in isolation, present some problem for the child.

Even though young children can be quite successful with smaller arrays, Zimiles found that they will not show even this limited capacity as readily on a problem that might shift their attention away from number. This suggests that the children have yet to develop complete confidence in their ability to work with even small numbers. Russell's (1936) data are in agreement with this hypothesis. Table X shows that variations in Russell's question form influenced performance. It can be seen that, in general, the larger the arrays, the greater the effect of the question variable. Still, even with small arrays, variations in the question form influenced performance, particularly for identical number combinations.

This discussion of confidence is particularly relevant to the next section on operators. As will be seen, a test of operational knowledge involves repeated questioning over time. It has already been suggested that children fail to pass conservation tests because they lack confidence in their initial response. In considering alternate tests of operational knowledge, it would seem advisable to take steps that might maximize the young child's confidence in his judgments.

In summary, young children may "fail" number tasks because they lack confidence in their ability to negotiate the tasks. However, this factor may be of less importance in tasks involving small numbers, reinforcing our earlier suggestion to focus on the use of small arrays in studies of young children's number operating abilities. Furthermore, in experiments in which it is desirable that children give a consistent performance, feedback techniques that buttress confidence can help serve this end.

IV. Operators

Thus far, attention has been concentrated on the young child's ability to abstract numerosity from an array of elements. At the beginning of the paper a distinction was made between operators and estimators, in part because it appears that the way a child estimates number may determine whether he shows competence for treating number operationally. It was suggested that whether or not a child behaves as if he has an invariance scheme for number will first depend on whether or not he estimates on the basis of number in a given situation. The research reported in this section was designed to assess whether children who estimate on the basis of number also possess number operators or a number invariance scheme. The presentation of this research is preceded by a section which expands the notion of operator that is used in this paper and provides a discussion of the nature of an operator test in conjunction with the relevant literature.

A. More about Operators

In general, the intervention of operators is inferred if an individual behaves as if he has considered an object or event in relation to the outcome of manipulations that can be or are performed on the object or event. With respect to number, we assume that operators mediate the ability to: (*a*) classify transformations of a set as relevant or irrelevant; (*b*) anticipate whether and how a particular manipulation will or will not change number; and (*c*) integrate perceptions from two or more successive presentations of a set (or sets) of elements through inferences about manipulations that change or do not change number.

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In general then, when an individual classifies events according to their effect on number, we postulate the use of operators. Further, we take as the fundamental feature of this classification the ability to distinguish between those events or manipulations which are relevant to number and those which are not. The presence of number operators not only leads to the classification of transformations and manipulations that are irrelevant: they also specify a complementary class of transformations and manipulations that are relevant to number, i.e., change number. Thus, for example, the operations of addition, subtraction, multiplication, and division belong to this complementary set because they are thought to change number. In contrast, the operations of rearrangement and displacement are classified as irrelevant because they are thought not to change number; the number of objects in a set is assumed to be invariant despite such manipulations.

As Piaget has pointed out, a mature concept of number may be mediated by higher order operations which specify the relations among lower order operators. For example, the inverse operator specifies that subtraction reverses addition and vice versa. The reason for postulating the inverse operator is that adults can anticipate the effect of addition followed by subtraction and vice versa and can specify the manipulation required to reverse some previous transformation.

Of the several manifestations of operators, the classification of transformations as relevant or irrelevant seems the most fundamental. Thus, the question of experimental interest is whether young children who estimate on the basis of number classify manipulations of sets as either relevant or irrelevant to number. Piaget (1957) and Flavell and Wohlwill (1969) acknowledged the possibility that very young children have some operators. Thus far, there has been no statement of what these might be and very little research that would help provide such a statement. The literature holds some suggestion that young children may understand the effects of addition and subtraction (see Section IV, B). But it is not known whether this understanding is part of an invariance scheme which classifies operations as relevant or irrelevant. One reason for the limited understanding in this area seems to be that tasks used to study operators are not suitable for preschool children. The following section takes up this problem in more detail.

B. Methodological Considerations

Implicit in the above discussion of operators is an assumption that an individual can or does think about number in terms of transformations that are performed over time. This specifies the nature of the class of behaviors that can be used to support the postulation of operators: Those that indicate

a subject is thinking about a given quantity at $time_2$ in terms of its status at $time_1$ and the events that have occurred in the interval. If, on a given test he behaves as if the events at $time_1$ and $time_2$ are independent, it cannot be determined whether he has taken into account the manipulations that have occurred in the interval. Thus, in order to infer operators one needs behavioral evidence that the subject is treating numerosity over time and in conjunction with what has happened or might have happened to the relevant set during the interval.

A consideration of Piaget's conservation tasks is in order here. They are the most widely used tests of the young child's quantity operators and appear to meet the above criterion. In the number conservation task a child is initially ($time_1$) shown two rows, one above the other, of N objects placed in one–one correspondence. The child is asked if the display represents an equivalence relationship with respect to number (e.g., "Does your row have as many as my row?"). Having acknowledged the numerical equivalence, the child watches as one row is transformed, e.g., lengthened, such that the perceptual relationship between the rows but not the numerical equivalence is altered. Then ($time_2$) the child is asked if numerical equivalence still holds. Thus, the task structure seemingly meets the criterion outlined above for the assessment of operators. In fact, the child's behavior may or may not be used to infer the presence of operators. He could treat the examples at $time_1$ and $time_2$ independently and take two estimates of the quantities. As has been argued above, the likelihood of his doing so may depend partly on the way he judges at $time_1$—i.e., whether he does it on the basis of number—and on his confidence in his ability to negotiate the task. Further, as other investigators have argued, he may misinterpret the questions (Braine, 1959) or be distracted by various features of the task (Bruner *et al.*, 1966; Gelman, 1969a; Wallach, Wall, & Anderson, 1967). All these considerations can be taken as forms of the general argument that the child might say that the quantities are different at $time_2$ because he takes the task to be two independent trials. If he does so, then we cannot decide whether or not he has operators. Failure to adopt the desired form of the task may or may not reflect his ability to operate on quantities. The suggestion of so many factors that could confound the assessment of young children's operative knowledge implies that the task may be too complex and that simpler procedures might better assess the question of interest. Before considering what these may be, there is a further issue regarding the conservation task.

What if the child does say the numbers are the same at $time_2$? Can the presence of operators be inferred unambiguously? It appears not. The child could still treat both examples of the relationship as independent estimation trials. Or the child may be stating that the elements of the sets

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are identical rather than asserting a belief in the continued equivalence of the sets (Elkind, 1967). Insofar as the interpretation of the judgments is ambiguous here, further criteria are necessary for the positive inference of operators. In this light, the reason for Piaget's insistence on verbal explanations becomes obvious. The child who says "They used to be the same," "You haven't added any," or "You just moved them around" is clearly thinking of the original displays and the manipulations performed on them. To the extent that a child can provide such statements, his data meet the criteria for inferring operators. Thus, good explanations become excellent data. The problem, however, is that the child must have sufficient verbal facility to give the investigator interpretable explanations. Young children may not be able to verbalize their understanding of the effect of transformations (Gelman & Weinberg, 1972). This simply adds to the variables already considered which confound the interpretation of the conservation task. It thereby lends support to the position that different operational tasks are needed for studies with younger children.

Since children who pass conservation tasks often offer the explanation that no amount has been added or subtracted since $time_1$, it can be concluded that these children possess addition and subtraction operators. The child who says that the amount is the same because neither addition nor subtraction has occurred is by implication telling us that he knows that the occurrence of these operations would indeed alter the quantity. Note that this type of assessment of addition and subtraction operators depends on the child's saying what has not occurred. It points to the possibility that children might better display their understanding of addition and subtraction operations in tasks which involve actual demonstrations of them. Several studies show that children can respond correctly to the effects of addition and subtraction before they conserve (e.g., Smedslund, 1966b; Wohlwill, 1960). In a study designed to permit a scalogram analysis of number concepts, Wohlwill included addition and subtraction as well as conservation items. The former involved adding or subtracting a button from a row of buttons in front of the child, scrambling the resulting row, and asking the child to match the result with one of three test cards. Children passed this test more readily than the conservation task, which involved merely scrambling. Smedslund analyzed the ability of kindergarten and first-grade children to take account of the effects of addition and subtraction operations. He also assessed the children's ability to conserve number. The addition/subtraction tasks involved establishing the equivalence of two sets placed side by side, covering the sets with boxes, and putting another element into one of the boxes or taking one out. After each addition or subtraction the child had to indicate whether both boxes had equal amounts or whether the left or right box had more. Each trial involved a series of

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three successive additions and/or subtractions performed on either the left or right box. Almost all children responded correctly to the first addition or subtraction on a three-item trial. That is, children could say which box had more after they witnessed either an addition or subtraction. In contrast, of 81 children, only 33 and 16, respectively, met the pre- and posttest conservation criteria of correct judgment and adequate explanation.

The conservation training literature provides further evidence that children can pass direct tests of addition and subtraction operators before they solve conservation problems. Various investigators have postulated that children who fail to conserve benefit from training that involves addition and subtraction (e.g., Smedslund, 1961, 1966b; Winer, 1968). These procedures do not always induce conservation and are open to alternative interpretations (Beilin, 1971); but it is still of interest that the nonconserver, as assessed by pretests, is able to respond accurately during the training phase. This implies that addition and subtraction tasks which involve the clear demonstration of these operations are easier for young children than is the conservation task. Thus, the evidence suggests that children who do not conserve still have some capacity to respond correctly to instances of operations of addition and subtraction.

The research on addition and subtraction operators has been limited to children of four years and older. In fact, most of the subjects in the reported studies are five and older. Could the previously employed paradigms be used with still younger children to determine the developmental course of such operators? A consideration of the techniques shows the inherent difficulties of doing so. These techniques typically involve a child's responding to a relationship between two arrays. Thus, for example, Smedslund's procedure requires the child to keep track of the nature of the quantitative relationship and to indicate whether two arrays are equal or if one is more (less) than the other after addition or subtraction has occurred. Rothenberg's (1969) work emphasizes the difficulty of such a procedure. The children are unlikely to use the critical words consistently. Some of Smedslund's writings (Smedslund, 1966b) reveal the same problem. Wohwill's procedure does not present the child with the semantic problem, but it involves a complex matching technique which frequently may be too difficult for very young children.

The foregoing considerations led to a search for a task that would not require young children to keep track of a quantitative relationship and yet would meet the criteria outlined above in the discussion of what constitutes a test of operators. The use of surprise reactions as an index of cognitive capacity (Achenbach, 1969; Charlesworth, 1969; Shantz & Watson, 1970) seemed promising. In surprise studies, it is assumed that a surprise reaction occurs when a subject's perception fails to conform with his expectations.

resumably, the expectation derives from and is carried over from previous xperiences that are related to the unexpected perception. If so, behaviors hich indicate a violation of expectancy might also indicate a subject's fforts to integrate past and present experiences in terms of events that pparently must have intervened to produce the discrepancy. To the extent hat this can be shown to be the case, reactions that reflect violations of xpectancy are in the class of behaviors that can be used to assess operators.

Magic shows suggest how to design a procedure that elicits surprise in order to study number operators. Magicians often perform tricks that involve violating invariance rules to produce surprise and mystification. Various reactions could be elicited if a magician worked with number. To begin with, he would have to establish an expectancy for a number of objects, just as he does for a particular colored scarf. Then he would begin covering and uncovering the objects. Sometimes he might simply rearrange them. In this case, the change should go unnoticed or should be explained in terms of an hypothesized intervening event (e.g., "The objects slipped") and/or should be said not to matter because it did not alter the number. On another occasion he might surreptitiously change the number. He then would wait for the audience to notice the change, express surprise, and even say that he must have done something to the original display. Occurrence of these differential reactions would show the audience's capacity to classify correctly operations that are relevant and irrelevant to number.

These considerations led the writer to develop a number "magic" task which involves two phases. In the first, the expectancy phase, the child is given experience with a set of objects. In the second phase, the child is shown a transformation of this set, one produced by a *surreptitious* addition, subtraction, or displacement. It was reasoned that number operators could be inferred if the children were surprised by transformations produced by addition and subtraction but not displacement, and if the children gave concomitant responses postulating the intervention of some unperceived event.

The procedure was deliberately designed to take into account the variables discussed throughout this paper. The procedure does not involve instructions that rely on the child's comprehension of quantity words or phrases such as "same" or "same number"; it involves feedback and the child's active participation in setting up trials in order to build the child's confidence in his understanding of the task; and it involves small numbers so as to maximize the likelihood of the child attending to numerosity.

C. Experimental Investigations of Operators

The studies reported here were designed first to determine whether young children have number invariance operators and second, whether the way

in which a child estimates number affects his tendency to use number operators. The magic paradigm briefly described above was used. In some experiments children were tested with very small arrays whose numerosity they should be able to estimate; in other experiments the arrays were large enough that the children might not be able to estimate on the basis of number. Two classes of surreptitious changes were introduced in Phase II of all experiments. These were (1) the addition or subtraction of one element, and (2) the lengthening or shortening of a row. The question is how children react to these different classes of operations as a function of the magnitude of an array. In particular, will they treat the effects of these operations as relevant or irrelevant to number when they can estimate on the basis of number but not when they cannot make such estimates?

1. *General Procedure of the Magic Experiment*

A detailed description of the procedure and design used in all of the following experiments can be found in Gelman (1972). In outline, the procedure involved two phases. The first built expectancies for two arrays of objects; the second assessed the child's reaction to surreptitious changes in one of the arrays. The expectancy phase began with showing a child two plates, each of which had a row of toy green mice. The number of mice on each plate differed (e.g., 2 vs. 3). The rows were either: (*a*) the same length, in which case a difference in density is redundant to the difference in number; or (*b*) the same density, in which case a length difference is redundant to the difference in number. While the child was looking at both plates and *without mentioning the difference in number*, the experimenter pointed to one of the plates (e.g., the 3-mouse plate) and said, "This plate is the winner." Then the child was told that he and the experimenter would take turns hiding the plates under two large cans and mixing them up. After each covering and shuffling, the child was asked to guess which can was hiding the winner plate. He lifted that can and answered the question "Is that the winner?" If the child chose correctly and confirmed his choice, he was told so and given a prize. A new trial of covering and shuffling then began. If, however, he selected the loser and said so, he was then asked where the winner was and was allowed to look under the remaining can. A correct identification at this point was reinforced. A new trial then began. Whenever a child erred in response to the question about whether he had uncovered a winner, he was corrected, the display was covered, and the can were reshuffled. Each lifting of a can was counted as a trial in order to control for total exposure of the arrays.[6] On three of the trials the child was asked why a plate is a winner or loser. The essential features of the

[6] Note that this was not a shell game and the child was not required to keep track of the winner. Indeed, shuffling was extensive to prevent this.

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procedure described so far were: (1) The child was *never* asked a question about *number* on the identification trials; (2) the child set up half the trials and therefore was involved directly in the game; (3) the child saw the equipment and both stimulus arrays, a condition designed to reduce the likelihood of children appealing to some unseen feature of the apparatus in Phase II; and (4) the child was given feedback in order to induce confidence in his judgments.

In most of the experiments, Phase II began after 10 or 11 trials (depending on whether the child uncovered a loser or winner on Trial 10). From the experimenter's point of view, this involved *covertly* changing the winning display either by adding or subtracting a mouse or lengthening or shortening the row. From the child's point of view, the onset of this phase was just another trial. The phase was brief. As soon as the child uncovered the altered display, be it on the first or second trial, he was asked if it was the winner. Overt surprise reactions were noted. The child was then asked why the display was a winner or loser; whether anything had happened, and if so, what; and how many mice were present now and before. Finally, he was asked if the game could be fixed and how. This was done to determine whether the children could undo or reverse any noticed changes.

2. *Studies with Two- and Three-Item Arrays*

In the estimation study described in Section II, D it was found that preschool children could consistently estimate on the basis of number when shown arrays with two and three items. Therefore, arrays of this magnitude were used in the first three "magic" studies. Two of these involved 3- to 6-year-old children and are reported elsewhere (Gelman, 1972). The third, with 2½-year-olds was done after the reported studies were completed. The first two studies are summarized here to provide continuity in the more detailed description of the subsequent research.

In all of the 2 vs. 3 studies the main design variable was the nature of the surreptitious change introduced in Phase II. Half the children in each experiment witnessed the effect of surreptitious subtraction or addition; the other half saw the effects of surreptitious displacement. The design also counterbalanced for the nature of the dimension (length or density) that was redundant in Phase I and for the locus (end, middle) of the addition or subtraction introduced or for whether the displacement consisted of lengthening or shortening.

a. *Experiment 1.* In Experiment 1, 32 children at each of three age levels (3, 4, and 5 years) were assigned to one of two of the basic conditions, subtraction or displacement. In Phase I, the 3-mouse array was designated the winner and the 2-mouse array the loser. The surreptitious operations were performed on the winner plate.

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The answers to probe questions in Phase I indicated that children chose number as opposed to length or density as the relevant attribute of the sets. Almost all children in each age group gave number descriptions of the plates either in response to probe questions in Phase I or as soon as they saw the altered array. In Phase II, 94% of the displacement children continued to identify the 3-mouse plate as the winner even though it was shorter or longer than it had been. In contrast, 69% of the subtraction children said the transformed plate lost; and the others either would not answer the question as to whether the uncovered plate was a winner or vacillated between saying "yes" or "no." These findings confirm the hypothesis that the children would abstract numbers of the magnitudes used. The critical question of interest is whether they reacted to Phase II as if they treated number as invariant.

This question is answered by analysis of the reactions and verbalizations for the particular transformation a child saw. Ninety-four percent of the subtraction subjects were judged to show surprise.[7] Only 27% of the rearrangement children were so judged. All children in the subtraction condition noticed the change and 80% searched. In contrast, only 52% of the displacement children noticed the change and none searched. The children's verbalizations reveal that these differential reactions indexed an understanding of what relevant or irrelevant operation must have intervened. Of the children in the displacement condition who did notice the change, almost all suggested that the mice moved. Still, they said they won because the plate had three mice on it. Thus, even though they noticed and accounted for changes in length or density, they acted as if these changes were irrelevant. In contrast, when subtraction children who appeared doubtful about winning were asked what had happened, they said the plate now had two mice instead of three. Further, the majority of these children went on to say that a mouse had been removed. Thus, the children behaved as if subtraction changed number but displacement did not.

Finally, an analysis of children's responses to questions about how the game might be fixed showed that 77% of the children knew that addition reversed subtraction or that elongation and shortening reversed each other. Thus, not only can subtraction and displacement operators be inferred, but so can higher order inverse operations.

b. *Experiment 2.* Children in Experiment 1 could reverse the effect of subtraction, suggesting that they knew that addition is an operation that is relevant to number. To provide direct evidence for this hypothesis the

[7] Criteria for scoring surprise, noticing, search, and verbalizations can be found in Gelman (1972). As indicated there, the scoring was highly reliable.

irst experiment was modified by simply designating the 2-mouse plate the winner and replacing the subtraction condition with an addition condition.

This experiment included 3- and 4-year-olds. Half the children were assigned to displacement conditions and half to addition conditions. Thus, in Phase II, children either saw the row of mice on the winner plate displaced or they saw an extra mouse on the winner plate.

The Phase I results were comparable to those reported for Experiment 1. The children identified the winner and loser on the basis of number. The Phase II results were also comparable. Surprise reactions occurred predominantly in the addition groups. All addition children noticed the change as opposed to only 38% of the displacement children. Children who noticed the changes could explain the nature of the operation that might have intervened; they postulated the operations of addition or displacement, depending on the condition they experienced. The children were also very likely to suggest or execute the relevant operation for undoing changes they noticed.

Together, Experiments 1 and 2 provide evidence that children as young as 3 years of age have number invariance operators and can use the operators on small numbers which they can estimate.

c. *Experiment 3.* Experiment 3 was the same as Experiment 1 except that the subjects were 2½-year-olds. Children were assigned to either a subtraction or displacement condition. The transformations were performed on the 3-mouse plate. A total of 21 subjects (median age, 2.7 years) was tested; 5 were excluded for failure to reach criterion in Phase I, leaving 8 children in each of the two basic conditions.

The reason for working with children of this age derives from Descoeudres' findings. As reported in Section II, B, Descoeudres tested the ability of 2½-year-olds to discriminate consistently between 2 and 3 item arrays on the basis of number. Only 26% of the 2½-year-olds could do so; whereas 41–67% of 3-year-olds and 67–100% of 4-year-olds passed this test. On the basis of this finding, it was expected that 2½-year-olds would be less likely than the children in Experiment 1 to use number operators. Descoeudres' finding can be interpreted to mean that 2½-years-olds are unable to estimate number on the basis of numerosity. As already indicated, a failure to estimate on the basis of number would predict a failure to use number operators.

The Phase I data from the children in these experiments confirmed Descoeudres' results in two ways. First, children in Experiments 1 and 2 seldom made an identification error in Phase I; they seldom failed to identify accurately an array as the winner or loser. When they did err, they always did so in the first half of the Phase I trials. In contrast, eight of the twenty-one 2½-year-olds in this experiment failed to reach a criterion of

5 correct out of 6; and five of these children failed to do so again on a second day and were dropped from the experiment. Second, only five of the sixteen remaining children could accurately describe the winner and/or loser with respect to number. One of the nine who failed to refer to number did talk about the winner having a "middle." The rest either gave irrelevant answers (e.g., "It just does") or no answer at all to probe questions.

The Phase II reactions of the 2½-year-olds were like those of the older children in some respects but not others. Seven of eight subtraction children said the altered array did not win or expressed doubts about its status, whereas 6 of 8 displacement children continued to identify the 3-mouse plate as the winner. Further, six subtraction children as opposed to one displacement child were surprised by the change.

The Phase II results differed from those reported for the older children in many ways. Only one subtraction child said there used to be three mice but there were only two now. None of the displacement children said they won because of the number. Of the 13 subjects who were judged to have noticed the change, only five made any kind of reference to an operation and none was able either to indicate how to undo the change or actually to undo it.

The five children who gave some indication of attributing the change to an operation were all subtraction subjects. They included three of the five children who used number in Phase I and the one who talked about the winner as having a middle. They were also the only subtraction children who searched in Phase II.

The data from the estimation phase of the experiment (Phase I) show that it is difficult for 2½-year-old children to estimate the numerosity of arrays of two and three objects. Some children had to be dropped from the experiment without proceeding to the test of their operators (Phase II).

The test of operators in those children who solved the estimation task yields ambiguous data. It cannot be said with any confidence that these children possess number invariance operators. However, those children who showed the strongest evidence of utilizing number invariance operators were the best estimators in Phase I. In other words the data suggest that behavioral manifestations of the use of number operators are closely tied to the ability to estimate the numbers involved accurately and confidently. The less accurate and the less confident the child's estimate of the numbers, the less evidence the child gives of using number invariance operators to integrate his perceptions of numerosity and manipulations of arrays.

It might be thought that the ambiguous behavior of the 2½-year-old children is attributable mostly to a general lack of verbal facility in children this young. Experiments with larger numbers and older children may indicate whether it is lack of verbal facility or lack of estimation ability that

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produces these results. Earlier research has shown that 3- and 4-year-old children have the verbal facility to give unequivocal evidence for the use of operators when shown arrays of two and three. If older children were found to respond to larger arrays as the 2½-year-olds did to 2 vs. 3 item arrays, the effect would seem to be attributable to number estimation difficulties rather than lack of verbal facility.

3. *Experiment 4: A Study with Three- and Four-Item Arrays*

Beckmann's and Descoeudres' findings indicate that 3- and 4-year-old children may estimate arrays as large as 3 items on the basis of number, but have difficulty doing so with 4-item arrays. This happened particularly in 3-year-olds. Experiment 4 consequently included 3- and 4-year-old groups and 3- and 4-item arrays. Equal numbers of children from each age level were assigned to either a displacement or an addition condition. The 3-mouse plate was designated the winner. In Phase II, the experimenter surreptitiously added 1 mouse to the 3-mouse plate or displaced (inward or outward) the original 3-mouse display.

In all, thirty-two 3-year-olds and twenty-two 4-year-olds were tested, but sixteen of the 3-year-olds and six of the 4-year-olds were dropped from the experiment for failure to reach Phase I criterion (correct identification of the winner). The median age of the subjects dropped was 3.3 and 4.1 years in the respective age groups. The median ages of the retained subjects was 3.5 and 4.4 years.

The need to drop some children indicates that the children in this experiment had difficulty learning to identify the winner and loser. It should be noted that before any children were dropped for failure to reach criterion, an attempt was made to train them in two stages. Stage I involved an extra 10 identification trials. Stage II involved what was thought might be an easy-to-difficult transfer situation (Lawrence, 1952). Children who failed to reach criterion after Stage I were brought back another day and tested on a 2 vs. 3 identification task. If they succeeded on this, they were once again given the 3 vs. 4 task for 10 or 11 trials. All children who succeeded after Stage I or II of extra training remained in the experiment; they were tested in Phase II as soon as they met the identification criterion. The training was effective to some extent, in that two of the 4-year-olds and nine of the 3-year-olds in the final sample had received some training.

The responses to probe questions in Phase I corroborated the impression that the much greater difficulty of the 3 vs. 4 task reflected the children's difficulty in estimating the numerosity of these larger arrays. With arrays of 2 vs. 3, 91% of the children in the previous experiments accurately described the winner and loser in terms of number. In the present experiment, only 62% of the 32 children in the final sample described the winner

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and loser in terms of number. Of the 10 who did not meet this criterion, 6 gave no answer to the probe questions or gave an irrelevant answer; 3 used number inaccurately (e.g., "One, two wins"); and 1 described the arrays with respect to their relative lengths. The difficulty with the identification task seems related to the ability to estimate on the basis of number. Of the 20 children who used number accurately in Phase I, only two needed extra training. Of the 12 who did not use number at all, or did so erroneously, 10 required extra training.

Table XI provides a breakdown of children in the addition and displacement conditions in terms of the type of response they gave in Phase I. Eleven children in each condition gave accurate responses to number and five children in each condition did not. For the children in the addition and displacement conditions who used number accurately in Phase I, the effects were like those reported for Experiment 2 (2 vs. 3 arrays). All 11 addition subjects said the altered array was a loser or expressed doubt on this point. When asked why they lost or were not sure, they all said that there were four mice and there should be three. Nine of 11 said either that there was an extra mouse or that a 4-mouse plate had been substituted for a 3-mouse plate. For example, one 3-year-old said "There's four, should be three. . . . We don't need this mouse. . . . Take one off and make three." As indicated in this protocol, these children could tell or show how the noticed transformation could be reversed. In contrast, all 11 displacement children who used number accurately in Phase I said they won. Nine said they did so because the array had 3 mice, although most of the children who seemed to notice the change (7 of 9) said that the mice had moved (together or apart). Finally, 7 of the 9 who noticed the change reversed it. These data, then, indicate that displacement children who estimated on the basis of number treated the operation of displacement as irrelevant to number. The protocol of D.L. (4.8) and E.B. (4.1) clearly illustrate this. In response to the question as to whether the transformed plate was the

TABLE XI

NUMBER OF CHILDREN IN ADDITION AND DISPLACEMENT CONDITIONS WHO DID OR DID NOT USE NUMBER IN PHASE I

	Phase II assignment	
Phase I response type	Addition	Displacement
Number: Accurate	11	11
Number: Inaccurate	1	2
Length	1	0
Other	3	3
	16	16

winner D.L. said, "Ya." ("How do you know?") "Cuz it has 3. . . . It's too far back cuz I saw; The first time I saw it closer than it is. Like this." ("How did that happen?") "I think we shook them . . . ." And E.B. said, "Even if it's mixed up it's still 3." ("How do you know?") "Cuz 1-2-3."

Several analyses of the Phase II reactions of the 10 children who did not use number at all in Phase I, or did so inaccurately, indicate that these children did not treat number operationally. First, none of the five addition children did not treat number operationally. First, none of the five addition children in this group was surprised by the transformation, whereas two of the displacement children were. Second, whether children in the addition condition thought they won seemed to depend on whether the length changed. The two children in this condition who said they had won were in a condition in which one mouse was added to the middle of the row; two of the three children in conditions which involved adding a mouse to the end of the row said they did not win. In fact, one of the latter children said he did not win because "Both are the same size." Third, although four of these five displacement children said the transformed plate was a winner, these children indicated that they thought length changes were relevant. For example, two children who saw shortened arrays said there were fewer mice: One 4-year-old said she was looking for another mouse and one 3-year-old said, "Maybe one went away." Fourth, of these 10 children, only one clearly suggested the intervention of an operation that was relevant to the condition he was in. Fifth, none of the children was able to reverse, even those who seemed to suggest irrelevant operations to account for perceived changes between Phase I and II. Finally, none of the children referred to numbers when asked why a display was the winner or loser in Phase II.

In sum, children who gave clear evidence of accurately estimating the numerosity of the initial arrays behaved as if they knew that the operation of addition and its reverse were relevant to number and that the operation of displacement was irrelevant. In contrast, children who failed to estimate on the basis of number or did so inaccurately failed to treat the operations of addition and subtraction as relevant or irrelevant to number. If anything, they may have behaved like Piaget's subjects who failed to conserve and treated changes in length as critical to number. The results of this study support the hypothesis that whether or not a young child estimates on the basis of number predicts whether or not he will reveal the use of number operators.

4. *Spontaneous Counting*

The protocols of children who treated number as invariant in the last experiment showed that a considerable number of them tended to count out loud spontaneously. This led to a consideration of the possible role of counting in the magic experiments.

As indicated in Section II, B, Beckmann found that young children count more than older children when estimating. Further, there is a greater tendency of children to count when confronted with large as opposed to small arrays. On the basis of these findings, children might be expected to count in order to estimate the number of mice on the winner and loser plates in Phase I. Moreover, more counting should occur in children shown 3 and 4 item arrays than in those shown 2 and 3 item arrays. Aside from serving as an estimator in Phase I, counting might also be the process by which children determine whether their expectancy for a given number has been violated. Children who notice transformations may count in order to classify them as irrelevant or relevant with respect to number or in order to confirm a decision to do so.

To assess these hypotheses, the protocols of children in Experiments 2 and 4 were examined. These experiments were selected for the analysis since the designs were the same with the sole exception that children in Experiment 2 saw 2 and 3 item arrays and children in Experiment 4 saw 3 and 4 item arrays. In both experiments, the covert operations of displacement or addition were performed on the array that contained fewer items. The age sample (3 and 4 years) was also comparable.

When children described arrays in terms of number, they either labeled or counted. Labeling involved the use of a number name ("It's got three"); counting involved the use of a sequence of number names ("One-two-three"). If a child who used number descriptions spontaneously counted out lound at least once within a phase, he was scored as a counter for that phase; otherwise he was scored as a labeler. A child who labeled and counted was scored as a counter for this analysis. Note that the assessment of the amount of counting is probably conservative in that a child was classified as a counter only if he counted out loud.

Table XII shows the proportions of children in Experiments 2 and 4

TABLE XII

PROPORTION OF CHILDREN IN EXPERIMENTS 2 AND 4 WHO USED NUMBER ACCURATELY AND EITHER LABELED OR COUNTED

Experiment	Type of Response	Phase I	Phase II
Experiment 2:	Count	.42	.65
2 vs.3	Label	.58	.35
Experiment 4:	Count	.64	.86
3 vs.4	Label	.36	.14

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who used number accurately and who were classified as counters or labelers in Phases I and II. A comparison of the proportion of children who counted at least once during Phase I of Experiments 2 and 4 confirms Beckmann's finding. There is a greater tendency to count given 3 vs. 4 item arrays than 2 vs. 3 item arrays.

Table XII also shows that children in both experiments were more likely to count in Phase II than in Phase I. This suggests that children do count especially when confronted with information that may constitute a discrepancy from an expectation. This result is relevant to the hypothesis of Wohlwill (1966) and Bearison (1969) that measurement operations mediate the development of conservation. As Bearison pointed out, the child who is able to measure quantities can then determine on his own whether or not a transformed quantity is conserved. Insofar as measurement involves counting, the present results are consistent with the positions of Wohlwill and Bearison.

V. Summary and Discussion

This paper has dealt with some of the variables that must be taken into account in order to understand the nature of number concepts in very young children. The distinction between operators and estimators served to organize the research and discussion of these variables. Particular emphasis has been placed on the way in which the child's ability to estimate accurately and with confidence interacts with his ability to understand quantity words, to attend to relevant or irrelevant features of a number task, and to use number operators. It was argued that one has to maximize the likelihood of a child's estimating on the basis of number in tasks designed to study operator processes. Given the finding that young children estimate very small numbers consistently, magic studies to assess whether young children have invariance operators for numbers in this range were conducted.

The results of the magic experiments show that, under some conditions, children as young as 3 years of age—and possibly 2½ years—behave as if they know that a small number of items in an array remains invariant through the operations of displacement but not addition and subtraction. Moreover, given the same range of numbers, children this young can understand the inverse relationship between addition and subtraction. Still, it appears that young children apply this invariance scheme only to small numbers. These findings raise two questions. The first is whether these children have a concept of number. The second concerns the role estimators play in the development of mature number abilities.

Do young children have a concept of number? They can abstract number from small arrays. They can infer that a change of number from 2 to 3 or vice versa must involve the addition of one item. They can classify the effects of a displacement operation as irrelevant to number. Surely, to some extent, they have a concept of number. These abilities indicate a complex categorization, one in which numbers are ordered, the numbers in a series are separated sequentially by a unit of 1, and sets that vary in length or density but contain the same number of units are classified in the same category. Moreover, the operators that treat addition and subtraction as relevant and displacement as irrelevant to number, and which assign an inverse relationship to addition and subtraction, constitute a number scheme that has *some* of the essential properties of axioms in the formal developments of arithmetic.

Despite these conclusions, the young child's notion of number has definite limits. One is that the invariance scheme applies only to numbers that are small enough for the child to estimate accurately and confidently. And, as reported elsewhere (Gelman, 1972), young children do not necessarily pass a number conservation task with three items. It has been suggested that the number conservation task is failed, not so much because the child lacks a number invariance scheme, but because he fails to understand the questions, lacks confidence, is readily misled, or shifts attention as he watches the transformations. But, since older children and adults do not have these difficulties with the task and are able, for example, to integrate their understanding of quantitative phrases with their number scheme, it still must be said that the number capacities of young children are limited. These, and other examples in Piaget's work, reveal the limits of the young child's application of the number scheme.

There seem to be two possible interpretations of these discrepancies. The child may be said to have a complete concept of number but to fail to use it in most situations because of his limited estimating abilities. He thus may have a number scheme which, in principle, could be applied to larger numbers. This position implies that the child knows there are numbers which he cannot estimate. Alternatively, the young child's number scheme may actually be incapable of dealing with numbers beyond a certain limit. The child may know only those numbers he can estimate, at least to some extent, and not recognize the existence of numbers beyond that limit. The writer is inclined to favor the second of these hypotheses. As Descoeudres reported, young children apparently fail to differentiate between numbers beyond a certain limit and instead classify them as "many."

Whichever interpretation is accepted, one faces the question of how the young child extends his concept of number to a wider range of situations. Our speculations on this issue derive from four observations about young children: (1) they know the effects of addition, subtraction, and displace-

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ment on a small set; (2) they know the specific effect of adding or subtracting one element to (from) small sets; (3) they tend to count when estimating numerosity; and (4) if they accurately estimate the numerosity of a set, they apply invariance operators to the same set. These abilities could serve as a base for the broader development of number. Children at a young age realize that 3 is generated from 2 by the addition of 1, and that 4 is generated from 3 by the addition of 1. This points the way to the realization that larger and larger numbers can be generated by the continued iteration of the addition of 1. Since the child seems to realize that his number scheme can be applied to what he counts, the realization that he could count indefinitely large numerosities should lead to the generalization of his number scheme. This is not to say that the child actually has to learn to count indefinitely large numbers. Rather, what he must grasp is the principle that numbers are constructed by the continued addition of units and are in principle countable. This position is congruent with that of Gal'perin and Georgiev (1969), who postulated that "The concept of the unit has a special place in the formation of elementary arithmetic. All other numbers are built from the unit (by the formula $n \pm 1$) . . . [p. 89]." It also reaches its most general expression in the principle of mathematical induction.

The above hypothesis suggests that an understanding of the development of estimators is central to an understanding of the development of number abilities. The ability to treat numbers operationally appears to depend on the ability to estimate numbers.

What has been suggested about the way children expand their use of a number invariance scheme to larger numbers has implications for the more general problem of how children form other quantity concepts. In principle, all quantities may be rendered countable through measurement. Thus, for example, the application of a ruler to a length is a process that translates length into a countable number of units. Similarly, liquid quantities can be measured with a cup. To the extent that quantities are rendered countable they may be handled by the number invariance scheme.

Such considerations suggest that the number scheme is a central quantity scheme which facilitates the development of other quantity concepts. Further, it may be that the processes which enable the child to apply the number scheme in general are essentially those which render quantities countable. Insofar as the logic underlying measurement hinges on the understanding that quantities can be translated into units, the problem in development becomes one of learning to measure (cf. Wohlwill, 1966). The finding that children come to conserve number before other quantities (Inhelder & Sinclair, 1969) is consistent with this suggestion. So is the fact that Bearison (1969) was able to teach children to conserve liquid quantities by teaching them to measure liquids.

As indicated at the beginning of the paper, the present approach to the analysis of the development of number concepts can, hopefully, be extended to other quantity concepts. On the basis of the foregoing, it can be suggested that insight into the nature of other quantity concepts may be gained from investigations of how children come to estimate or measure quantities and whether the operators children use to deal with other quantities are comparable or identical to the operators in their number scheme.

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